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Spectral and Ergodic Properties of the Analytic Generators*

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In this paper similarity situations between one-parameter groups of operators are characterized in terms of analytic generators and spectral subspaces, and ergodic properties of one-parameter groups are studied. These topics permit us to give "noncommutative" extensions of classical results from the theory of Hardy's H^p spaces. As major applications we mention the possibility of use of the analyticity notion from Arveson (*Amer. J. Math.* **89** (1967), 578–642) for one-parameter groups of $*$-automorphisms of operator algebras with "good" ergodic properties (see Section 3), and a procedure to "improve" implementing groups of unitaries for one-parameter groups of $*$-automorphisms of operator algebras (see Section 5).

INTRODUCTION

Let X be a Banach space, σ an appropriate weak topology on X and $\{U_t\}_{t\in\mathbb{R}}$ a bounded σ-continuous one-parameter group of σ-continuous linear operators on X. Following [8] we can associate to $\{U_t\}$ an injective σ-closed linear operator B in X, called the analytic generator of $\{U_t\}$, which determines uniquely $\{U_t\}$. In [8] and in the first section of this paper, spectral subspaces of B are associated to each closed connected subset of $(0, +\infty)$, in the same way as this is done for injective positive selfadjoint linear operators in Hilbert spaces. It is shown that these spectral subspaces are the same as those defined in [3].

In Section 2 we characterize general similarity situations between one-parameter groups in terms of their analytic generators and in terms of the associated spectral subspaces (Theorems 2.1 and 2.2).

In particular, the spectral subspaces of B determine uniquely $\{U_t\}$. Part of these results was proved under some additional assumptions in [3] and our proofs are strongly inspired from the techniques used in [3]. Theorem 2.1 can

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be used in a treatment of the basic facts in Tomita's theory of standard von Neumann algebras, following the sketch given in [28]. Theorem 2.2 implies general implementation results (Corollaries 2.5 and 2.6) which are used later.

In Section 3 we analyze the influence of a "weak ergodic property" of $\{U_t\}$ on the density properties of the spectral subspaces (Theorem 3.1 and Corollary 3.2) and establish a connection between an "ergodic property" of $\{U_t\}$ and the behavior of B^n when $n \to +\infty$ or $n \to -\infty$ (Theorem 3.6). If X is a Banach algebra with a separately σ-continuous product, U_t are multiplicative and $\{U_t\}$ have the "global ergodic property, "then we can associate to $\{U_t\}$ an "analyticity structure" in the sense of [1] (Theorem 3.8 and the discussion at the end of this section). This is an answer to a question raised in [3].

In Section 4 we give an imbedding of $\{U_t\}$ in a one-parameter group which acts in the dual of a Banach space and which is $*$-weakly continuous. Using an extension of the classical F. and M. Riesz theorem (Theorem 4.1) we characterize this imbedding in the case when X is a C^*-algebra with a separately σ-continuous product and U_t are $*$-automorphisms (Corollary 4.2). We remark that Theorem 4.1 was proved under a norm-continuity hypothesis with essentialy the same proof, in [3]. Part of the classical Szegö–Kolmogorov–Krein theorem is also extended (Theorem 4.3).

In Section 5 we consider a von Neumann algebra $X \subset B(H)$ and an X_*-continuous one-parameter group $\{U_t\}$ of $*$-automorphisms of X and we describe those closed linear subspaces of H which are invariant under the action of the operators belonging to the spectral subspace of B associated to $(0, 1]$ (Theorems 5.1, 5.2, and 5.3). So we obtain extensions of clasical invariant space results of Wiener, Beurling, Lax, Helson, and Lowdenslager (for these we send you to [14]). We also obtain an extension of Wermer's Maximality Theorem (Corollary 5.9).

Theorem 5.3 permits us to point out a procedure to "improve" implementing groups of unitaries for $\{U_t\}$. Using this procedure, we reobtain the proof presented in [3] for some implementation results of Kadison, Sakai, and Borchers (Corollaries 5.6 and 5.7) and we also obtain another implementation result (Theorem 5.12).

We remark that some results contained in this paper (for exemple, Theorem 2.2 and Lemma 5.10) can be extended for general locally compact groups of operators (see [29]).

1. Spectral Subspaces

The aim of this section is to complete the results of [3, Section 2; 8, Section 5].

In this paper, by a *dual pair of Banach spaces* we understand any pair $(X, \mathscr{F})$ of complex Banach spaces, together with a bilinear functional:

$$X \times \mathscr{F} \ni (x, \varphi) \mapsto \langle x, \varphi \rangle,$$

such that:

(i) $$\|x\| = \sup_{\substack{\varphi \in \mathscr{F} \\ \|\varphi\| \leqslant 1}} |\langle x, \varphi \rangle| \qquad \text{for any} \quad x \in X;$$

(ii) $$\|\varphi\| = \sup_{\substack{x \in X \\ \|x\| \leqslant 1}} |\langle x, \varphi \rangle| \qquad \text{for any} \quad \varphi \in \mathscr{F};$$

(iii) the convex hull of each relatively $\mathscr{F}$-compact subset of X is relatively $\mathscr{F}$-compact:

(iv) the convex hull of each relatively X-compact subset of $\mathscr{F}$ is relatively X-compact.

We recall that in [8] the definition of a dual pair of Banach spaces requires only the present conditions (i) and (ii). However, in the main results from [8] also conditions (iii) and (iv) are required.

We remark that if X is an arbitrary Banach space, then (X, X^*) and (X^*, X) are dual pairs of Banach spaces.

Let Ω be a locally compact Hausdorff space, μ a complex regular Borel measure on Ω, and $F : \Omega \to X$ such that for each $\varphi \in \mathscr{F}$ the function:

$$\Omega \ni \alpha \to \langle F(\alpha), \varphi \rangle$$

is μ-integrable. If there exists $x_F \in X$ such that

$$\int_\Omega \langle F(\alpha), \varphi \rangle \, d\mu(\alpha) = \langle x_F, \varphi \rangle, \qquad \varphi \in \mathscr{F},$$

then we denote

$$x_F = \mathscr{F} - \int_\Omega F(\alpha) \, d\mu(\alpha).$$

Sufficient conditions for the existence of the *$\mathscr{F}$-integral* are given by Proposition 1.2 [3] and Proposition 1.4 [8].

Denote by $B_{\mathscr{F}}(X)$ the linear space of all $\mathscr{F}$-continuous linear operators on X. A *one-parameter group* $\{U_t\}_{t \in \mathbb{R}}$ in $B_{\mathscr{F}}(X)$ is called *$\mathscr{F}$-continuous* if for each $x \in X$ and $\varphi \in \mathscr{F}$ the function $t \mapsto \langle U_t x, \varphi \rangle$ is continuous. We say that $\{U_t\}$ is *bounded* if $\sup_{t \in \mathbb{R}} \|U_t\| < +\infty$.

If Ω is a subset of $\mathbb{C}$ and $F ; \Omega \to X$, then F is called $\mathscr{F}$-regular if it is $\mathscr{F}$-continuous and its restriction to the interior of Ω is analytic. For any $\mathscr{F}$-

continuous one-parameter group $\{U_t\}$ in $B_{\mathscr{F}}(X)$ and any $\alpha \in \mathbb{C}$, a linear operator B_α is defined (see [8], Section 2):

$$\mathscr{D}_{B_\alpha} = \left\{ x \in X; \begin{array}{l} it \to U_t x \text{ has an } \mathscr{F}\text{-regular extension } F_x \text{ on} \\ \text{the closed vertical strip between 0 and Re}\,\alpha \end{array} \right\}$$

$$B_\alpha x = F_x(\alpha), \qquad x \in \mathscr{D}_{B_\alpha}.$$

B_α is called the *analytical extension* of $\{U_t\}$ in α. The *analytic generator* of $\{U_t\}$ is $B = B_1$. By [8], Lemma 2.2, the sequential $\mathscr{F}$-closure of $\bigcup_{\alpha \in C} \mathscr{D}_{B_\alpha}$ is X.

If T is an $\mathscr{F}$-densely defined linear operator in X then we define the *adjoint* T^* of T in $\mathscr{F}$ in a usual way:

$$\mathscr{D}_{T^*} = \left\{ \varphi \in \mathscr{F}; \begin{array}{l} \text{there exists } \psi \in \mathscr{F} \text{ such that for each } x \in \mathscr{D}_T \\ \langle x, \psi \rangle = \langle Tx, \varphi \rangle \end{array} \right\}$$

$$T^*\varphi = \psi, \qquad \varphi \in \mathscr{D}_{T^*}.$$

It is easy to see that T^* is X-closed and if $T \in B_{\mathscr{F}}(X)$ then $T^* \in B_X(\mathscr{F})$. If $\{U_t\}$ is an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ then $\{U_t^*\}$ is an X-continuous one-parameter group in $B_X(\mathscr{F})$.

1.1. THEOREM. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t \in \mathbb{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B_α its analytical extension in $\alpha \in \mathbb{C}$. Then the analytical extension of $\{U_t^*\}_{t \in \mathbb{R}}$ in α is the adjoint B_α^* of B_α.*

Proof. Let ψ be an element in the domain of the analytical extension of $\{U_t^*\}$ in α. Then there exists an X-regular mapping $F_\varphi : \{\gamma \in \mathbb{C}$: Re γ between 0 and Re $\alpha\} \to \mathscr{F}$ such that

$$F_\varphi(it) = U_t^*\varphi, \qquad t \in \mathbb{R}.$$

For each $x \in \mathscr{D}_{B_\alpha}$ the regular functions

$$\gamma \mapsto \langle x, F_\varphi(\gamma) \rangle,$$

$$\gamma \mapsto \langle B_\gamma x, \varphi \rangle.$$

defined on $\{\gamma \in \mathbb{C}$: Re γ between 0 and Re $\alpha\}$, coincide on the imaginary axis, so they coincide everywhere. In particular,

$$\langle x, F_\varphi(\alpha) \rangle = \langle B_\alpha x, \varphi \rangle, \qquad x \in \mathscr{D}_{B_\alpha}.$$

Hence $\varphi \in \mathscr{D}_{B_\alpha^*}$ and $B_\alpha^* \varphi$ is equal with the value in φ of the analytical extension of $\{U_t^*\}$ in α.

Now let $\varphi \in \mathscr{D}_{B_\alpha^*}$.

By [15], Theorem 7.6.1 (see also [8], Lemma 2.1) there exist $a, c > 0$ such that

$$\| U_t \| \leq c \cdot e^{a \cdot |t|}, \qquad t \in \mathbb{R}.$$

Let $x \in X$ be arbitrary and consider a net $\{x\iota\} \subset \mathscr{D}_{B_\alpha}$ which converge to x in the Mackey topology associated to the $\mathscr{F}$-topology. For each ι

$$\gamma \mapsto e^{-(\gamma/i)^2} \langle B_\gamma x\iota, \varphi \rangle$$

is a bounded regular function on $\{\gamma \in \mathbb{C};\ \mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha\}$. Since the convex hull of

$$\{e^{-t^2} \cdot U_t^* \varphi;\ t \in \mathbb{R}\} \subset \mathscr{F}$$

is relatively X-compact

$$e^{-t^2} \langle B_{it} x_\iota, \varphi \rangle = e^{-t^2} \langle x\iota, U_t^* \varphi \rangle \to e^{-t^2} \langle x, U_t^* \varphi \rangle$$

uniformly for $t \in \mathbb{R}$. Analogously,

$$\begin{aligned} &e^{-((\alpha+it)/i)^2} \langle B_{\alpha+it} x\iota, \varphi \rangle \\ &\quad = e^{-((\alpha+it)/i)^2} \langle x\iota, U_t^* B_\alpha^* \varphi \rangle \to e^{-((\alpha+it)/i)^2} \langle x, U_t^* B_\alpha^* \varphi \rangle \end{aligned}$$

uniformly for $t \in \mathbb{R}$. By the Pragmén–Lindelöf principle, the net $\{\gamma \mapsto e^{-(\gamma/i)^2} \langle B_\gamma x\iota, \varphi \rangle\}$ of bounded regular functions on $\{\gamma \in \mathbb{C};\ \mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha\}$ converges uniformly to a bounded regular function G_x. For each $t \in \mathbb{R}$

$$G_x(it) = e^{-t^2} \langle x, U_t^* \varphi \rangle,$$

$$G_x(\alpha + it) = e^{-((\alpha+it)/i)^2} \langle x, U_t^* B_\alpha^* \varphi \rangle.$$

In particular, G_x does not depend on the choice of the net $\{x\iota\}$. Denoting by K the convex hull of

$$\{e^{-t^2} U_{t\varphi}^*,\ e^{-((\alpha+it)/i)^2} U_t^* B_\alpha^* \varphi;\ t \in \mathbb{R}\} \subset \mathscr{F},$$

K is relatively X-compact. For any $x \in X$ and any $\gamma \in \mathbb{C}$ with $\mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha$

$$|G_x(\gamma)| \leq \sup_{\psi \in K} |\langle x, \psi \rangle|.$$

So, for fixed $\gamma \in \mathbb{C}$ with $\mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha$, $x \to G_x(\gamma)$ is a linear functional on X, continuous in the Mackey topology associated to the $\mathscr{F}$-topology. Hence there exists $G(\gamma) \in \mathscr{F}$ such that

$$\langle x, G(\gamma) \rangle = G_x(\gamma), \qquad x \in X.$$

Obviously, G is an X-regular mapping defined on $\{\gamma \in \mathbb{C}\colon \operatorname{Re} \gamma$ between 0 and $\operatorname{Re} \alpha\}$ and

$$G(it) = e^{t^2} U_t{}^* \varphi, \qquad t \in \mathbb{R},$$

$$G(\alpha) = e^{-(\alpha/i)^2} B_\alpha{}^* \varphi.$$

Putting $F(\gamma) = e^{(\gamma/i)^2} G(\gamma)$, F is an X-regular mapping on $\{\gamma \in \mathbb{C}; \operatorname{Re} \gamma$ between 0 and $\operatorname{Re} \alpha\}$ and

$$F(it) = U_t{}^* \varphi, \qquad t \in \mathbb{R},$$

$$F(\alpha) = B_\alpha{}^* \varphi.$$

Consequently φ is in the domain of the analytical extension of $\{U_t{}^*\}$ in α and the value of this in φ is $B_\alpha{}^*\varphi$. Q.E.D.

In particular, the analytic generator of $\{U_t{}^*\}$ is the adjoint of the analytic generator of $\{U_t\}$.

We remark that by Theorem 1.1 $B_\alpha = B_\alpha^{**}$, so B_α is $\mathscr{F}$-closed (see [8], Theorem 2.4). Theorem 1.1 implies also the selfadjointness of the analytic generator of one-parameter groups of unitaries on a Hilbert space (see [8], Theorem 6.1).

If $\{U_t\}$ is a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B is its analytic generator, then for $0 < \lambda_1 \leqslant \lambda_2 < +\infty$ the spectral subspace $X^B([\lambda_1, \lambda_2])$ is defined by

$$X^B([\lambda_1, \lambda_2]) = \left\{ x \in \bigcap_{n=-\infty}^{+\infty} \mathscr{D}_{B^n} ;\ \overline{\lim_{n \to +\infty}} \| B^n x \|^{1/n} \leqslant \lambda_2 ,\ \overline{\lim_{n \to +\infty}} \| B^{-n} x \|^{1/n} \leqslant \frac{1}{\lambda_1} \right\}.$$

We recall that these spectral subspaces satisfy properties (i)–(vii) from [8], Section 5. By property (v) and Theorem 5.2 [8], the restriction of $\{U_t\}$ to each $X^B([\lambda_1, \lambda_2])$, $0 < \lambda_1 \leqslant \lambda_2 < +\infty$, is uniformly continuous. Now we define for $0 < \lambda < +\infty$ the spectral subspaces

$$X^B((0, \lambda]) = \left\{ x \in \bigcap_{n=1}^{+\infty} \mathscr{D}_{B^n} ;\ \overline{\lim_{n \to +\infty}} \| B^n x \|^{1/n} \leqslant \lambda \right\},$$

$$X^B([\lambda, +\infty)) = \left\{ x \in \bigcap_{n=-\infty}^{-1} \mathscr{D}_{B^n} ;\ \overline{\lim_{n \to +\infty}} \| B^{-n} x \|^{1/n} \leqslant \frac{1}{\lambda} \right\}.$$

By the proof of [8], Lemma 5.1

$$\| B_\epsilon x \| \leqslant \lambda^\epsilon \sup_{t\in\mathbb{R}} \| U_t \| \cdot \| x \|, \qquad x \in X^B((0, \lambda]), \quad \epsilon > 0,$$

$$\| B_{-\epsilon} x \| \leqslant \lambda^{-\epsilon} \sup_{t\in\mathbb{R}} \| U_t \| \cdot \| x \|, \qquad x \in X^B([\lambda, +\infty)), \quad \epsilon > 0.$$

In particular, $X^B((0, \lambda])$ and $X^B([\lambda, +\infty))$ are norm-closed.

1.2. LEMMA. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_α its analytical extension in $\alpha \in \mathbb{C}$ and $0 < \lambda < +\infty$. Then for any $x \in X^B((0, \lambda])$*

$$B_\alpha x = \frac{\lambda^\alpha}{\pi} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{\operatorname{Re}\alpha}{(\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha - t)^2} \lambda^{-it} U_t x \, dt, \qquad \operatorname{Re}\alpha > 0,$$

and for each $x \in X^B([\lambda, +\infty))$

$$B_\alpha x = -\frac{\lambda^\alpha}{\pi} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{\operatorname{Re}\alpha}{(\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha - t)^2} \lambda^{-it} U_t x \, dt, \qquad \operatorname{Re}\alpha < 0.$$

Proof. Let $x \in X^B((0, \lambda])$. Define the bounded $\mathscr{F}$-regular mapping F on $\{\alpha \in \mathbb{C}: \operatorname{Re}\alpha \geqslant 0\}$ by

$$F(\alpha) = \lambda^{-\alpha} B_\alpha x.$$

Using the Poisson formula (see [16, Chap. 8.]), we have

$$F(\alpha) = \frac{1}{\pi} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{\operatorname{Re}\alpha}{(\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha - t)^2} F(it) \, dt, \qquad \operatorname{Re}\alpha > 0,$$

so

$$B_\alpha x = \frac{\lambda^\alpha}{\pi} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{\operatorname{Re}\alpha}{(\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha - t)^2} \lambda^{-it} U_t x \, dt, \qquad \operatorname{Re}\alpha > 0.$$

The proof of the second formula is analogous. Q.E.D.

By Lemma 1.2 B_α, $\operatorname{Re}\alpha \geqslant 0$, are $\mathscr{F}$-continuous on $X^B((0, \lambda])$, $0 < \lambda < +\infty$, and B_α, $\operatorname{Re}\alpha \leqslant 0$, are $\mathscr{F}$-continuous on $X^B([\lambda, +\infty))$, $0 < \lambda < +\infty$.

1.3. LEMMA. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and $0 < \lambda < +\infty$. Then*

$$X^B((0, \lambda]) = \textit{sequential } \mathscr{F}\textit{-closure of } \bigcup_{0<\mu\leqslant\lambda} X^B([\mu, \lambda]),$$

$$X^B([\lambda, +\infty)) = \textit{sequential } \mathscr{F}\textit{-closure of } \bigcup_{\lambda\leqslant\mu<+\infty} X^B([\lambda, \mu]).$$

Proof. Let $\{x_k\} \subset X^B((0, \lambda])$ be an $\mathscr{F}$-convergent sequence and x its limit. By [8], Proposition 1.1, $\sup_k \|x_k\| < +\infty$.

Let $n \geq 1$. Following Lemma 1.2, for each k

$$B^n x_k = \frac{\lambda^n}{\pi} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{n}{n^2+t^2} \lambda^{-it} U_t x_k \, dt.$$

Using the Lebesgue dominated convergence theorem, we deduce that the sequence $\{B^n x_k\}_k$ is convergent in the $\mathscr{F}$-topology. Since B^n is $\mathscr{F}$-closed (see [8], Theorem 2.4), it follows that

$$x \in \mathscr{D}_{B^n}, \qquad B^n x = \mathscr{F} - \lim_k B^n x_k .$$

So

$$\|B^n x\| \leq \lambda^n \sup_{t\in\mathbb{R}} \|U_t\| \sup_k \|x_k\|.$$

Since $n \geq 1$ is arbitrary, it follows that

$$x \in \bigcap_{n=1}^{+\infty} \mathscr{D}_{B^n}, \qquad \lim_{n\to+\infty} \|B^n x\|^{1/n} \leq \lambda,$$

that is

$$x \in X^B((0, \lambda]).$$

Consequently $X^B((0, \lambda])$ is sequentially $\mathscr{F}$-closed and we deduce that sequential $\mathscr{F}$-closure of $\bigcup_{0<\mu\leq\lambda} X^B([\mu, \lambda]) \subset X^B((0, \lambda])$. Now let $x \in X^B((0, \lambda])$. Consider a sequence $\{f_k\} \subset L^1(\mathbb{R})$ as in [8], Lemma 5.5. Using [8], Corollary 2.5, it is easy to see that for any k

$$x_k = \mathscr{F} - \int_{-\infty}^{+\infty} f_k(t) \, U_t x \, dt \in X^B((0, \lambda]).$$

On the other hand, by [8], Lemma 5.4, $x_k \in X^B([e^{-3k}, e^{3k}])$. So, for $e^{-3k} \leq \lambda$,

$$x_k \in X^B([e^{-3k}, \lambda]).$$

Finally, following [8], Lemma 5.5, $\mathscr{F}\text{-}\lim_k x_k = x$. Consequently,

$$X^B((0, \lambda]) \subset \text{sequential } \mathscr{F}\text{-closure of } \bigcup_{0<\mu\leq\lambda} X^B([\mu, \lambda]).$$

The proof of the second equality is similar. Q.E.D.

For each $f \in L^1(\mathbb{R})$ we denote by $\hat{f}$ its inverse Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t) \, e^{ist} \, dt.$$

We define for $0 < \lambda < +\infty$

$$X^B((0, \lambda)) = \text{sequential } \mathscr{F}\text{-closure of}$$
$$\times \left\{ \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt;\ \begin{array}{l} x \in X, f \in L^1(\mathbb{R}), \hat{f} \in C^2(\mathbb{R}) \\ \text{supp}\, \hat{f} \text{ is compact and is} \\ \text{included in } (-\infty, \ln \lambda) \end{array} \right\},$$

$$X^B((\lambda, +\infty)) = \text{sequential } \mathscr{F}\text{-closure of}$$
$$\times \left\{ \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt;\ \begin{array}{l} x \in X, f \in L^1(\mathbb{R}), \hat{f} \in C^2(\mathbb{R}) \\ \text{supp}\, \hat{f} \text{ is compact and is} \\ \text{included in } (\ln \lambda, +\infty) \end{array} \right\}.$$

We shall prove now the "regularity property" corresponding to [3], Proposition 2, and [8], Theorem 5.6;

1.4. THEOREM. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and $0 < \lambda < +\infty$. Then*

$$X^B((0, \lambda]) = \bigcap_{\delta>1} X^B((0, \lambda\delta))$$
$$= \left\{ x \in X;\ \begin{array}{l} \textit{for each } f \in L^1(\mathbb{R}), \text{supp}\, \hat{f} \cap (-\infty, \ln \lambda] = \varnothing \\ \textit{we have } \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt = 0 \end{array} \right\};$$

and

$$X^B([\lambda, +\infty)) = \bigcap_{\delta>1} X^B((\lambda\delta^{-1}, +\infty))$$
$$= \left\{ x \in X;\ \begin{array}{l} \textit{for each } f \in L^1(\mathbb{R}), \text{supp}\, \hat{f} \cap [\ln \lambda, +\infty) = \varnothing \\ \textit{we have } \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt = 0 \end{array} \right\}.$$

Proof. Denote

$$S = \left\{ x \in X;\ \begin{array}{l} \text{for each } f \in L^1(\mathbb{R}), \text{supp}\, \hat{f} \cap (-\infty, \ln \lambda] = \varnothing \\ \text{we have } \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt = 0 \end{array} \right\}.$$

Then S in $\mathscr{F}$-closed.

By [8], Corollary 5.7, for any $0 < \mu \leqslant \lambda$

$$X^B([\mu, \lambda]) \subset S,$$

so, using Lemma 1.3,

$$X^B((0, \lambda]) \subset S.$$

Let $x \in S$ and $\delta > 1$. Consider a sequence $\{f_n\} \subset L^1(\mathbb{R})$ as in [8], Lemma 5.5, and denote

$$x_n = \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)\, U_t x\, dt.$$

If $f \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{f} \cap [-3n, \ln \lambda] = \emptyset$, then $\operatorname{supp}(f * f_n) \cap (-\infty, \ln \lambda] = \emptyset$, so

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x_n\, dt = \mathscr{F} - \int_{-\infty}^{+\infty} (f * f_n)\, U_t x\, dt = 0.$$

Using [8], Theorem 5.6, it follows that

$$x_n \in X^B([e^{-3k}, \lambda]) \subset X^B((0, \lambda\delta)).$$

Hence

$$x = \mathscr{F} - \lim_n x_n \in X^B((0, \lambda\delta)).$$

Consequently,

$$S \subset \bigcap_{\delta>1} X^B((0, \lambda\delta)).$$

Finally, let $\delta > 1$ and $x \in X((0, \lambda\delta))$. Then there exists a sequence $\{f_n\} \subset L^1(\mathbb{R})$ such that for each n, $f_n \in C^2(\mathbb{R})$, $\operatorname{supp} \hat{f}_n \subset (\ln \mu_n, \ln(\lambda\delta))$ where $0 < \mu_n \leqslant \lambda\delta$, and a sequence $\{x_n\} \subset X$ such that

$$x = \mathscr{F} - \lim_n \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)\, U_t x_n\, dt.$$

Using [8], Theorem 5.6, it is easy to see that

$$\mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)\, U_t x_n\, dt \in X^B([\mu_n, \lambda\delta]).$$

By Lemma 1.3 it follows that $x \in X^B((0, \lambda\delta])$. Thus

$$\bigcap_{\delta>1} X^B((0, \lambda\delta)) \subset \bigcap_{\delta>1} X^B((0, \lambda\delta]) = X^B((0, \lambda]).$$

The formulas for $X^B([\lambda, +\infty))$ have analogous proofs. Q.E.D.

An important consequence of Theorem 1.4 is that each $X^B((0, \lambda])$ and $X^B([\lambda, +\infty))$ is closed. So, property (vii) from [8], Section 5, is conserved for the spectral subspaces introduced here.

We remark that by Theorem 1.4 Arveson's spectral subspaces $M^U((-\infty, \ln \lambda])$ and $M^U([\ln \lambda, +\infty))$ from [3] coincide with $X^B((0, \lambda])$, respectively, $X^B([\lambda, +\infty))$.

If $0 < \lambda < +\infty$, in general X is not the $\mathscr{F}$-closure of $X^B((0, \lambda]) + X^B([\lambda, +\infty))$, so the statement before [3], Proposition 2.2, is not true. For example, if $X = C([0, 1])$, $\mathscr{F} = X^*$ and

$$(U_t f)(s) = e^{its} f(s), \qquad f \in X,$$

then it is easy to see that

$$X^B((0, e^{1/2}]) = \{f \in X: \operatorname{supp} f \subset [0, \tfrac{1}{2}]\},$$

$$X^B([e^{1/2}, +\infty)) = \{f \in X: \operatorname{supp} f \subset [\tfrac{1}{2}, 1]\},$$

so, for each f in the $\mathscr{F}$-closure of $X^B((0, e^{1/2}]) + X^B([e^{1/2}, +\infty))$ we have $f(\tfrac{1}{2}) = 0$.

In Section 3 we shall see that the above statement is related with the ergodic properties of the group $\{\lambda^{-it} U_t\}$.

We shall now give the connection between the spectral subspaces of $\{U_t\}$ and $\{U_t^*\}$.

For $S \subset \mathscr{F}$ we denote by $S_X^\perp$ its annulator in X;

$$S_X = \{x \in X: \langle x, \varphi\rangle = 0 \text{ for any } \varphi \in S\}.$$

1.5. Corollary. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t \in 1k}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{2k}(X)$ and $0 < \lambda < +\infty$. Then*

$$X^B((0, \lambda]) = \Big(\bigcup_{\lambda < \mu < +\infty} \mathscr{F}^{B^*}([\mu, +\infty))\Big)_X^\perp$$

and

$$X^B([\lambda, +\infty)) = \Big(\bigcup_{0 < \mu < \lambda} \mathscr{F}^{B^*}((0, \mu])\Big)_X^\perp .$$

Proof. Using Theorem 1.4, it is easy to see that both $X^B((0, \lambda])$ and $(\bigcup_{\lambda < \mu \leqslant +\infty} \mathscr{F}^{B^*}([\mu, +\infty)))_X^\perp$ are equal with the set of all $x \in X$ for which

$$\int_{-\infty}^{+\infty} f(t) \langle U_t x, \varphi\rangle \, dt = 0$$

whenever $f \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{f} \subset (\ln \lambda, +\infty)$ and $\varphi \in \mathscr{F}$.

The proof of the second equality is similar. Q.E.D.

Finally, supposing that in X we have additional structures compatible with the duality and that $\{U_t\}$ preserves these structures, we look for special properties of the spectral subspaces.

We say that a complex Banach space $\mathscr{F}$ is *in duality with a complex Banach algebra* X if there exists a bilinear functional

$$X \times \mathscr{F} \ni (x, \varphi) \mapsto \langle x, \varphi \rangle$$

with which $(X, \mathscr{F})$ becomes a dual pair of Banach spaces and for each $x \in X$ the mappings

$$y \to xy$$

$$y \to yx$$

are $\mathscr{F}$-continuous. In this case, for each $\varphi \in \mathscr{F}$ and $x \in X$ there exist elements $L_x\varphi$ and $R_x\varphi$ of $\mathscr{F}$ such that

$$\langle y, L_x\varphi \rangle = \langle xy, \varphi \rangle, \qquad y \in X,$$

$$\langle y, R_x\varphi \rangle = \langle yx, \varphi \rangle, \qquad y \in X.$$

We say that a complex Banach space $\mathscr{F}$ is *in duality with a complex Banach space with involution* X if there exists a bilinear functional

$$X \times \mathscr{F} \ni (x, \varphi) \mapsto \langle x, \varphi \rangle$$

with which $(X, \mathscr{F})$ becomes a dual pair of Banach spaces and the mapping

$$y \mapsto y^*$$

is $\mathscr{F}$-continuous. In this case, for each $\varphi \in \mathscr{F}$ there exists $\varphi^* \in \mathscr{F}$ such that

$$\langle y, \varphi^* \rangle = \overline{\langle y^*, \varphi \rangle}, \qquad y \in X.$$

Now let X be a C^*-algebra. Suppose that $\mathscr{F}$ is in duality with the Banach algebra X: then $\mathscr{F}$ can be considered a subspace of X^* and X^{**} is a W^*-algebra with predual X^* (see [24], Theorem 1.17.2). So, for $\varphi \in \mathscr{F}$ and $x \in X^{**}$, one can define the elements $L_x\varphi$, $R_x\varphi$ of X^* and it is easy to verify that $L_x\varphi$ and $R_x\varphi$ belong to $\mathscr{F}$. If $\varphi \in \mathscr{F}$ and $\varphi = R_v |\varphi|$ is its polar decomposition (see [24], Theorem 1.14.4) then $|\varphi| = R_{v^*}\varphi \in \mathscr{F}$ hence, putting $\varphi^* = L_{v^*} |\varphi|$, we have $\varphi^* \in \mathscr{F}$ and

$$\langle y, \varphi^* \rangle = \langle v^* y, |\varphi| \rangle = \overline{\langle y^* v, |\varphi| \rangle} = \overline{\langle y^*, \varphi \rangle}, \qquad y \in Y.$$

It follows that $\mathscr{F}$ is in duality also with the Banach space with involution X.

We remark that if X is a C^*-algebra, then X^* is in duality with X and if X is a W^*-algebra, then also its predual X_* is in duality with X.

1.6. THEOREM. *Let X be a complex Banach algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous automorphisms of the algebra X. Then*:

(i) *for each $\alpha \in \mathbb{C}$, $\mathscr{D}_{B_\alpha}$ is a subalgebra of X and B_α is multiplicative*;

(ii) *for each $0 < \lambda, \mu < +\infty$,*

$$X^B((0, \lambda])\, X^B((0, \mu]) \subset X^B((0, \lambda\mu]),$$
$$X^B([\lambda, +\infty))\, X^B([\mu, +\infty)) \subset X^B([\lambda\mu, +\infty)).$$

Further, let X be a complex Banach space with involution, $\mathscr{F}$ a Banach space in duality with X and $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous automorphisms of the linear space with involution X. Then:

(i) *for each $\alpha \in \mathbb{C}$, $(\mathscr{D}_{B_\alpha})^* = \mathscr{D}_{B_{-\bar\alpha}}$ and*

$$B_{-\bar\alpha}x^* = (B_\alpha x)^*, \qquad x \in \mathscr{D}_{B_\alpha};$$

(ii) *for each $0 < \lambda < +\infty$,*

$$X^B((0, \lambda])^* = X^B([1/\lambda, +\infty)).$$

Proof. We suppose first that X is a Banach algebra and U_t are multiplicative. Let $0 < \lambda_1 \leqslant \lambda_2 < +\infty$, $x \in X^B([\lambda_1, \lambda_2])$ and $0 < \mu_1 \leqslant \mu_2 < +\infty$, $y \in X^B([\mu_1, \mu_2])$. Then $\alpha \mapsto (B_\alpha x)(B_\alpha y)$ is an integer extension of $it \mapsto (U_t x)(U_t y) = U_t(xy)$, so

$$xy \in \bigcap_{\alpha\in\mathbb{C}} \mathscr{D}_{B_\alpha} \qquad B_\alpha(xy) = (B_\alpha x)(B_\alpha y), \qquad \alpha \in \mathbb{C}.$$

Moreover,

$$\overline{\lim_{n\to\infty}} \| B^n(xy)\|^{1/n} \leqslant \overline{\lim_{n\to\infty}} (\| B^n x \|^{1/n} \| B^n y \|^{1/n}) \leqslant \lambda_2\mu_2,$$
$$\overline{\lim_{n\to\infty}} \| B^{-n}(xy)\|^{1/n} \leqslant \overline{\lim_{n\to\infty}} (\| B^{-n} x \|^{1/n} \| B^{-n} y \|^{1/n}) \leqslant \frac{1}{\lambda_1\mu_1}$$

that is

$$xy \in X^B([\lambda_1\mu_1, \lambda_2\mu_2]).$$

Using the fact that for each $\alpha \in \mathbb{C}$ the $\mathscr{F}$-closure of $B_\alpha \mid \bigcup_{0<\lambda_1\leqslant\lambda_2<+\infty} X^B([\lambda_1, \lambda_2])$ is B_α, it follows (i) and using Lemma 1.3, (ii) follows.

Second, we suppose that X is a Banach space with involution and that U_t commute with the involution. Let $\alpha \in \mathbb{C}$ and $x \in \mathscr{D}_{B_\alpha}$; then $\gamma \mapsto (B_{-\bar\gamma} y)^*$ is an $\mathscr{F}$-regular extension of $it \mapsto (U_t x)^* = U_t x^*$ to $\{\gamma \in \mathbb{C}\colon \operatorname{Re}\gamma$ between 0 and $\operatorname{Re}(-\bar\alpha)\}$, so

$$x^* \in \mathscr{D}_{B_{-\bar\alpha}}, \qquad B_{-\bar\alpha}x^* = (B_{-\overline{(-\bar\alpha)}}x)^* = (B_\alpha x)^*.$$

If $0 < \lambda < +\infty$ and $x \in X^B((0, \lambda])$ then

$$x^* \in \bigcap_{n=1}^{\infty} (\mathscr{D}_{B^n})^* = \bigcap_{n=-\infty}^{-1} \mathscr{D}_{B^n},$$

$$\overline{\lim_{n\to\infty}} \| B^{-n}x \|^{1/n} = \overline{\lim_{n\to\infty}} \|(B^n x)^* \|^{1/n} \leq \lambda,$$

that is

$$x \in X^B([1/\lambda, +\infty)).$$

The proof of the inclusion $X^B([1/\lambda, +\infty)) \subset X^B((0, \lambda])^*$ is analogous.

Q.E.D.

We remark that Theorem 1.6(ii) extends [3], Lemma 3.1. Theorem 1.6(ii) can be proved easily also if we start with Arveson's definition of the spectral subspaces and it was known by Arveson.

Suppose that X is a Banach algebra and U_t are multiplicative. By Theorem 1.6(ii), $X^B((0, 1])$ is a subalgebra of X and $X^B((0, \lambda])$, $0 < \lambda < 1$, are two-sided ideals of $X^B((0, 1])$. Analogously, $X^B([1, +\infty))$ is a subalgebra of X and $X([\lambda, +\infty))$, $1 < \lambda < +\infty$, are two-sided ideals of $X^B([1, +\infty))$. In Section 3, we shall give conditions under which there exists an $\{U_t\}$-invariant $\mathscr{F}$-continuous linear projection of X onto $X^B([1, 1])$ which is multiplicative on $X^B((0, 1])$ and on $X^B([1, +\infty))$, so obtaining an answer to a question raised at the end of [3].

We remark also that in Section 2 we give a strong extension of Theorem 1.6.

2. Similarity Results

In this section we characterize the similarity of two bounded continuous one-parameter groups in terms of their analytic generators and in terms of the associated spectral subspaces. We prove also a general implementation theorem.

We recall that if $(X, \mathscr{F})$ is a dual pair of Banach spaces, $\{U_t\}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, and B its analytic generator, then for any $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ the operator $\lambda + B$ is injective and its image contains $\mathscr{D}_B$ (see [8], Section 3).

2.1. Theorem. *Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be dual pairs of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, $\{V_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{G}$-continuous one-parameter group in $B_{\mathscr{G}}(Y)$, B and D their analytic generators, and Φ a linear operator from X into Y, whose graph is closed in the product of the $\mathscr{F}$-topology with the $\mathscr{G}$-topology.*

Then the following statements are equivalent:

(i) *for each* $x \in \mathscr{D}_\Phi$ *and* $t \in \mathbb{R}$,

$$U_t x \in \mathscr{D}_\Phi, \qquad \Phi U_t x = V_t \Phi x;$$

(ii) *the closure of the graph of the restriction of* Φ *to*

$$\left\{x \in X;\ \begin{array}{l} x \in \mathscr{D}_B \cap \mathscr{D}_\Phi,\ (\lambda + B)^{-1} x \in \mathscr{D}_\Phi,\ \Phi x \in \mathscr{D}_D \\ \text{and } \Phi(\lambda + B)^{-1} x = (\lambda + D)^{-1} \Phi x \text{ for all } \lambda > 0 \end{array}\right\}$$

in the product of the $\mathscr{F}$*-topology with the* $\mathscr{G}$*-topology is the graph of* Φ;

(iii) *for each* $0 < \lambda_1 \leqslant \lambda_2 < +\infty$

$$U_t(X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi) \subset \mathscr{D}_\Phi, \qquad t \in \mathbb{R},$$

$\mathbb{R} \ni t \to \Phi U_t x$ *is bounded and* $\mathscr{G}$*-continuous for* $x \in X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi$,

$$\Phi(X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi) \subset Y^D([\lambda_1, \lambda_2]),$$

and the closure of the graph of the restriction of Φ *to*

$$\bigcup_{0 < \lambda_1 \leqslant \lambda_2 < +\infty} (X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi)$$

in the product of the $\mathscr{F}$*-topology with the* $\mathscr{G}$*-topology is the graph of* Φ.

Proof. Suppose that (i) is verified. Then for each $x \in \mathscr{D}_\Phi$ and $f \in L^1(\mathbb{R})$ we have

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt \in \mathscr{D}_\Phi$$

and

$$\Phi\left(\mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt\right) = \mathscr{G} - \int_{-\infty}^{+\infty} f(t)\, V_t \Phi(x)\, dt.$$

Using [8], Corollary 5.7, it follows that

$$\Phi(X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi) \subset Y^D([\lambda_1, \lambda_2]), \quad 0 < \lambda_1 \leqslant \lambda_2 < +\infty.$$

Let $f \in L^1(\mathbb{R})$ such that $\hat{f} \in C^2(\mathbb{R})$ and $\operatorname{supp} \hat{f} \subset [\ln \lambda_1, \ln \lambda_2]$, $0 < \lambda_1 \leqslant \lambda_2 < +\infty$: then f has an integer extension which is Lebesgue integrable on each horizontal line. For each $x \in X$ and $\alpha \in \mathbb{C}$, we have:

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt \in X^B([\lambda_1, \lambda_2])$$

and

$$B_\alpha\left(\mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x\, dt\right) = \mathscr{F} - \int_{-\infty}^{+\infty} f(t + i\alpha)\, U_t x\, dt.$$

So, for $x\in\mathscr{D}_\Phi$ and $\alpha\in\mathbb{C}$, we have

$$B_\alpha\left(\mathscr{F}-\int_{-\infty}^{+\infty} f(t)\,U_t x\,dt\right)\in\mathscr{D}_\Phi,\qquad \Phi\left(\mathscr{F}-\int_{-\infty}^{+\infty} f(t)\,U_t x\,dt\right)\in\mathscr{D}_{D_\alpha}$$

and

$$\Phi B_\alpha\left(\mathscr{F}-\int_{-\infty}^{+\infty} f(t)\,U_t x\,dt\right)=D_\alpha\Phi\left(\mathscr{F}-\int_{-\infty}^{+\infty} f(t)\,U_t x\,dt\right).$$

Using [8], Lemma 5.5, it follows that the closure of the graph of the restriction of Φ to

$$\bigcup_{0<\lambda_1\leqslant\lambda_2<+\infty}\{x\in X^B([\lambda_1,\lambda_2]);\ B_\alpha x\in\mathscr{D}_\Phi,\ \Phi x\in\mathscr{D}_{D_\alpha},\ \Phi B_\alpha x=D_\alpha\Phi x \text{ for all } \alpha\}$$

in the product of the $\mathscr{F}$-topology with the $\mathscr{G}$-topology, is the graph of Φ.

Now it is easy to see that (iii) is verified. Using [8], Corollary 3.3, it is also easy to verify (ii).

Consequently we have the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Let us suppose now that (ii) is verified and denote by:

$$\mathscr{D}=\left\{x\in X;\ \begin{matrix} x\in\mathscr{D}_B\cap\mathscr{D}_\Phi,\ (\lambda+B)^{-1}x\in\mathscr{D}_\Phi,\ x\in\mathscr{D}_D \text{ and}\\ \Phi(\lambda+B)^{-1}x=(\lambda+D)^{-1}\Phi x \text{ for all } \lambda>0\end{matrix}\right\}.$$

Let $x\in\mathscr{D}$: then for all $\lambda>0$

$$(\lambda+B)^{-1}Bx=x-\lambda(\lambda+B)^{-1}x\in\mathscr{D}_\Phi,$$
$$\Phi(\lambda+B)^{-1}Bx=\Phi x-\lambda(\lambda+D)^{-1}\Phi x=(\lambda+D)^{-1}D\Phi x.$$

Using [8], Theorem 4.2, it follows that for each $\alpha\in\mathbb{C}$, $0<\operatorname{Re}\alpha<1$,

$$B_\alpha x\in\mathscr{D}_\Phi,\qquad \Phi B_\alpha x=D_\alpha\Phi x.$$

Consequently, for each $t\in\mathbb{R}$,

$$U_t x\in\mathscr{D}_\Phi,\qquad \Phi U_t x=V_t\Phi x.$$

Since the closure of the graph of $\Phi\,|\,\mathscr{D}$ in the product of the $\mathscr{F}$-topology with the $\mathscr{G}$-topology is the graph of Φ, it follows that (i) is verified.

So we proved that (ii) $\Rightarrow$ (i).

Let us finally suppose that (iii) is verified.

Let $0<\lambda_1\leqslant\lambda_2<+\infty$, $x\in X([\lambda_1,\lambda_2])\cap\mathscr{D}_\Phi$ and $0<\mu_1\leqslant\mu_2<+\infty$, $\varphi\in\mathscr{G}^{D^*}([\mu_1,\mu_2])$: then $t\to\Phi U_t x$ is bounded and $\mathscr{G}$-continuous while $s\to V_s^*\varphi$ is bounded and norm-continuous. In particular, the function

$$(t,s)\to\langle V_s\Phi U_t x,\varphi\rangle=\langle\Phi U_t x,V_s^*\varphi\rangle$$

is bounded and continuous.

Let $f \in L^1(\mathbb{R})$ be such that $0 \notin \operatorname{supp} \hat{f}$. Then there exists $\epsilon > 0$ such that $[-\epsilon, \epsilon] \cap \operatorname{supp} \hat{f} = \varnothing$. If $g \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g} \subset [-\epsilon, \epsilon]$, then, using the inversion formula for Fourier transforms, it is easy to see that g is equal almost everywhere with an uniformly continuous bounded function.

For each $g \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g} \subset [-\epsilon, \epsilon]$, and $r \in \mathbb{R}$

$$\mathscr{F} = \int_{-\infty}^{+\infty} g(t)\, e^{irt} U_t x \, dt \in X^B([e^{-\epsilon-r}, e^{\epsilon-r}]) \cap \mathscr{S}_\Phi ,$$

$$\Phi\left(\mathscr{F} = \int_{-\infty}^{+\infty} g(t)\, e^{irt} U_t x\, dt\right) = \mathscr{G} = \int_{-\infty}^{+\infty} g(t)\, e^{irt} \Phi U_t x\, dt \in Y^D([e^{-\epsilon-r}, e^{\epsilon-r}]),$$

so we have successively

$$\mathscr{G} = \int_{-\infty}^{+\infty} f(s)\, e^{irs} V_s \left(\mathscr{G} = \int_{-\infty}^{+\infty} g(t)\, e^{irt} \Phi U_t x\, dt\right) ds = 0,$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ir(t+s)} g(t) f(s) \langle V_s \Phi U_t x, \varphi \rangle \, dt\, ds = 0,$$

$$\int_{-\infty}^{+\infty} e^{irt} \left(\int_{-\infty}^{+\infty} g(t-s) f(s) \langle V_s \Phi U_{t-s} x, \varphi \rangle \, ds \right) dt = 0,$$

Consequently, for each $g \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g} \subset [-\epsilon, \epsilon]$, the continuous Lebesgue integrable function

$$t \to \int_{-\infty}^{+\infty} g(t-s) f(s) \langle V_s \Phi U_{t-s} x, \varphi \rangle \, ds$$

vanishes identically, hence

$$\int_{-\infty}^{+\infty} g(-s) f(s) \langle V_s \Phi U_{-s} x, \varphi \rangle \, ds = 0.$$

But if $g \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g} \subset [-\epsilon, \epsilon]$, then each translate $g_t = g(t + \cdot)$ verifies: $\operatorname{supp} \hat{g}_t \subset [-\epsilon, \epsilon]$. It follows that for each $g \in L^1(\mathbb{R})$ with supp $\hat{g} \subset [-\epsilon, \epsilon]$ and $t \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} g(t-s) f(s) \langle V_s \Phi U_{-s} x, \varphi \rangle \, ds = 0.$$

So, denoting

$$h(s) = f(s) \langle V_s \Phi U_{-s} x, \varphi \rangle,$$

we have $g * h = 0$, that is $\hat{g} \cdot \hat{h} = 0$, for each $g \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g} \subset [-\epsilon, \epsilon]$.

Consequently, $\hat{h}$ vanishes on $[-\epsilon, \epsilon]$. In particular,

$$\int_{-\infty}^{+\infty} f(s)\langle V_s \Phi U_{-s} x, \varphi\rangle\, ds = \hat{h}(0) = 0,$$

that is

$$\int_{-\infty}^{+\infty} f(-s)\langle V_{-s} \Phi U_s x, \varphi\rangle\, ds = 0.$$

Now, if $f \in L^1(\mathbb{R})$, $0 \notin \operatorname{supp} \hat{f}$, then for each translate $f_t = f(t - \cdot)$, we have $0 \notin \operatorname{supp} \hat{f}_t$. Thus for each $f \in L^1(\mathbb{R})$, $0 \notin \operatorname{supp} \hat{f}$, and $t \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} f(t - s)\langle V_{-s} \Phi U_s x, \varphi\rangle\, ds = 0.$$

Denoting

$$k(s) = \langle V_{-s} \Phi U_s x, \varphi\rangle,$$

$\mathscr{T} = \{f \in L^1(\mathbb{R}) : f * k = 0\}$ is a closed ideal in the convolution algebra $L^1(\mathbb{R})$ and contains all $f \in L^1(\mathbb{R})$ such that $0 \notin \operatorname{supp} \hat{f}$.

Hence hull $\mathscr{T} \subset \{0\}$ and by Shilov's theorem (see [19], 37 C)

$$\mathscr{T} = \ker(\operatorname{hull} \mathscr{T}) \subset \{f \in L^1(\mathbb{R}) : \hat{f}(0) = 0\}.$$

Consequently, the kernel of the functional

$$L^1(\mathbb{R}) \ni f \mapsto \int_{-\infty}^{+\infty} f(-s)\, k(s)\, ds$$

contains the kernel of the functional

$$L^1(\mathbb{R}) \ni f \mapsto \int_{-\infty}^{+\infty} f(-s)\, ds = \hat{f}(0),$$

so k is constant. This means that for each $s \in \mathbb{R}$

$$\langle V_{-s} \Phi U_s x, \varphi\rangle = \langle \Phi x, \varphi\rangle.$$

Since $\bigcup_{0<\mu_1\leqslant\mu_2<+\infty} \mathscr{G}^{D^*}([\mu_1, \mu_2])$ is Y-dense in $\mathscr{G}$, we deduce that for each $x \in X^B([\lambda_1, \lambda_2]) \cap \mathscr{D}_\Phi$, $0 < \lambda_1 \leqslant \lambda_2 < +\infty$, and $s \in \mathbb{R}$

$$V_{-s} \Phi U_s x = \Phi x, \qquad \Phi U_s x = V_s \Phi x.$$

Finally, since the closure of the graph of $\Phi \mid \bigcup_{0<\lambda_1\leqslant\lambda_2<+\infty} X^B([\lambda_1, \lambda_2])$ in the product of the $\mathscr{F}$-topology with the $\mathscr{G}$-topology is the graph of Φ, it follows that for each $x \in \mathscr{D}_\Phi$ and $s \in \mathbb{R}$

$$U_s x \in \mathscr{D}_\Phi, \qquad \Phi U_s x = V_s \Phi x.$$

So we have proved that (iii) $\Rightarrow$ (i). Q.E.D.

The equivalence of statements (i) and (ii) in Theorem 2.1 was suggested to us by the proof of Tomita's fundamental theorem in the theory of standard von Neumann algebras sketched in [28].

The equivalence of statements (i) and (iii) in Theorem 2.1 was proved in [3] for the case of everywhere defined Φ and in the presence of assumptions like [3], Hypothesis 1.5(iii). Our proof is strongly inspired from Arveson's method.

We remark that Theorem 2.1 can be used in a treatment of the basic facts in Tomita's theory of standard von Neumann algebras.

We shall prove now another similarity result which can be considered as an implementation theorem.

2.2. THEOREM. *Let $(X^1, \mathscr{F}^1),..., (X^n, \mathscr{F}^n), (Y^1, \mathscr{G}^1),..., (Y^m, \mathscr{G}^m)$, $(Z, \mathscr{H})$ be dual pairs of Banach spaces, $\{U_t^i\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}^i$-continuous one-parameter group in $B_{\mathscr{F}^i}(X^i)$, B_α^i its analytical extension in $\alpha \in \mathbb{C}$ and B^i its analytic generator, $1 \leqslant i \leqslant n$, $\{V_t^j\}_{t\in\mathbb{R}}$ a bounded $\mathscr{G}^j$-continuous one-parameter group in $B_{\mathscr{G}^j}(Y^j)$, D_α^j its analytical extension in $\alpha \in \mathbb{C}$ and D^j its analytic generator, $1 \leqslant j \leqslant m$ and $\{W_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{H}$-continuous one-parameter group in $B_{\mathscr{H}}(Z)$, E_α its analytical extension in $\alpha \in \mathbb{C}$ and E its analytic generator. Consider a mapping*

$$\Phi: X^1 \times \cdots \times X^n \times Y^1 \times \cdots \times Y^m \to Z$$

such that for each $1 \leqslant i \leqslant n$ and $x^1,..., x^{i-1}, x^{i+1},..., x^n, y^1,..., y^m$,

$$X^i \ni x^i \to \Phi(x^1,..., x^n, y^1,..., y^m)$$

is linear and continuous for the $\mathscr{F}^i$-topology on X^i and the $\mathscr{H}$-topology on Z, for each $1 \leqslant j \leqslant m$ and $x^1,..., x^n, y^1,..., y^{j-1}, y^{j+1},..., y^m$,

$$Y^j \ni y^j \mapsto \Phi(x^1,..., x^n, y^1,..., y^m)$$

is antilinear and continuous for the $\mathscr{G}^j$-topology on Y^j and the $\mathscr{H}$-topology on Z and the multiadditive mapping Φ is bounded. Then the following statements are equivalent:

(i) *for each $x^1,..., x^n, y^1,..., y^m$ and $t \in \mathbb{R}$*

$$\Phi(U_t^1x^1,..., U_t^nx^n, V_t^1y^1,..., V_t^my^m) = W_t\Phi(x^1,..., x^n, y^1,..., y^m);$$

(ii) *for each $\alpha \in \mathbb{C}$ and $x^1 \in \mathscr{D}_{B_\alpha^1},..., x^n \in \mathscr{D}_{B_\alpha^n}, y^1 \in \mathscr{D}_{D^1_{-\bar\alpha}},..., y^m \in \mathscr{D}_{D^m_{-\bar\alpha}}$ we have $\Phi(x^1,..., x^n, y^1,..., y^m) \in \mathscr{D}_{E_\alpha}$ and*

$$\Phi(B_\alpha^1x^1,..., B_\alpha^nx^n, D^1_{-\bar\alpha}y^1,..., D^m_{-\bar\alpha}y^m) = E_\alpha\Phi(x^1,..., x^n, y^1,..., y^m);$$

(iii) *for each* $x^1 \in \mathscr{D}_{B^1}, \ldots, x^n \in \mathscr{D}_{B^n}$, $y^1 \in \mathscr{D}_{(D^1)^{-1}}, \ldots, y^m \in \mathscr{D}_{(D^m)^{-1}}$ *we have* $\Phi(x^1, \ldots, x^n, y^1, \ldots, y^m) \in \mathscr{D}_E$ *and*

$$\Phi(B^1x^1, \ldots, B^nx^n, (D^1)^{-1} y^1, \ldots, (D^m)^{-1} y^m) = E\Phi(x^1, \ldots, x^n, y^1, \ldots, y^m)$$

(iv) *for each* $0 < \lambda_1{}^i \leqslant \lambda_2{}^i < +\infty$, $1 \leqslant i \leqslant n$, and $0 < \mu_1{}^j \leqslant \mu_2{}^j < +\infty$, $1 \leqslant j \leqslant m$,

$$\Phi((X^1)^{B^1}([\lambda_1{}^1, \lambda_2{}^1]), \ldots, (X^n)^{B^n}([\lambda_1{}^n, \lambda_2{}^n]), (Y^1)^{D^1}([\mu_1{}^1, \mu_2{}^1]), \ldots, (Y^m)^{D^m}([\mu_1{}^m, \mu_2{}^m]))$$

$$\subset Z^E\left(\left[\frac{\lambda_1{}^1 \cdots \lambda_1{}^n}{\mu_2{}^1 \cdots \mu_2{}^m}, \frac{\lambda_2{}^1 \cdots \lambda_2{}^n}{\mu_1{}^1 \cdots \mu_1{}^m}\right]\right).$$

Proof. Suppose that (i) is verified. Let $0 < \lambda_1{}^i \leqslant \lambda_2{}^i < +\infty$, $x^i \in (X^i)^{B^i}([\lambda_1{}^i, \lambda_2{}^i])$, $1 \leqslant i \leqslant n$ and $0 < \mu_1{}^j \leqslant \mu_2{}^j < +\infty$, $y^j \in (Y^j)^{D^j}([\mu_1{}^j, \mu_2{}^j])$. Then

$$\alpha \to \Phi(B_\alpha{}^1 x^1, \ldots, B_\alpha{}^n x^n, D^1_{-\bar{\alpha}} y^1, \ldots, D^m_{-\bar{\alpha}} y^m)$$

is an integral extension of $it \to W_t\Phi(x^1, \ldots, x^n, y^1, \ldots, y^m)$, so

$$\Phi(x^1, \ldots, x^n, y^1, \ldots, y^m) \subset \bigcap_{\alpha \in \mathbf{C}} \mathscr{D}_{E_\alpha},$$

$$E_\alpha\Phi(x^1, \ldots, x^n, y^1, \ldots, y^m) = \Phi(B_\alpha{}^1 x^1, \ldots, B_\alpha{}^n x^n, D^1_{-\bar{\alpha}} y^1, \ldots, D^m_{-\bar{\alpha}} y^m), \quad \alpha \in \mathbb{C}.$$

Using the fact that for each $\alpha \in \mathbb{C}$ the $\mathscr{F}^i$-closure of

$$B_\alpha{}^i \Big| \bigcup_{0 < \lambda_1{}^i \leqslant \lambda_2{}^i < +\infty} (X^i)^{B^i}([\lambda_1{}^i, \lambda_2{}^i])$$

is $B_\alpha{}^i$, $1 \leqslant i \leqslant n$, and the $\mathscr{G}^j$-closure of

$$D^j_{-\bar{\alpha}} \Big| \bigcup_{0 < \mu_1{}^j \leqslant \mu_2{}^j < +\infty} (Y^j)^{D^j}([\mu_1{}^j, \mu_2{}^j])$$

is $D^j_{-\bar{\alpha}}$, $1 \leqslant j \leqslant m$, it follows (ii).

So we have proved that (i) $\Rightarrow$ (ii); the implication (ii) $\Rightarrow$ (iii) is trivial.

Now we suppose that (iii) is verified.

Let be $x^i \in \mathscr{D}_{(B^i)^{-1}}$, $1 \leqslant i \leqslant n$, and $y^j \in \mathscr{D}_{D^j}$, $1 \leqslant j \leqslant m$. We have successively

$$(B^i)^{-1} x^i \in \mathscr{D}_{B^i}, \qquad D^j y^j \in \mathscr{D}_{(D^j)^{-1}}, \qquad \Phi((B^1)^{-1} x^1, \ldots, D^m y^m) \in \mathscr{D}_E$$

and

$$\Phi(x^1, \ldots, y^m) = \Phi(B^1(B^1)^{-1} x^1, \ldots, (D^m)^{-1} D^m y^m) = E\Phi((B^1)^{-1} x^1, \ldots, D^m y^m),$$

$$\Phi(x^1, \ldots, y^m) \in \mathscr{D}_{E^{-1}} \quad \text{and} \quad \Phi((B^1)^{-1} x^1, \ldots, D^m y^m) = E^{-1}\Phi(x^1, \ldots, y^m).$$

Now it is easy to see that if $x^i \in \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(B^i)^k}$, $1 \leqslant i \leqslant n$, and $y^j \in \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(D^j)^k}$, $1 \leqslant j \leqslant m$, then $\Phi(x^1,..., y^m) \in \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{E^k}$ and

$$\Phi((B^1)^k x^1,..., (B^n)^k x^n, (D^1)^{-k} y^1,..., (D^m)^{-k} y^m) = E^k\Phi(x^1,..., y^m), \ k \in \mathbb{Z}.$$

Consequently, if $0 < \lambda_1{}^i \leqslant \lambda_2{}^i < +\infty$, $x^i \in (X^i)^{B^i}([\lambda_1{}^i, \lambda_2{}^i])$, $1 \leqslant i \leqslant n$, and $0 < \mu_1{}^j \leqslant \mu_2{}^j < +\infty$, $y^j \in (Y^j)^{D^j}([\mu_1{}^j, \mu_2{}^j])$, $1 \leqslant j \leqslant m$, then $\Phi(x^1,..., y^m) \in \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{E^k}$ and

$$\overline{\lim_{k\to\infty}} \| E^k\Phi(x^1,..., y^m)\|^{1/k} \leqslant \overline{\lim_{k\to\infty}} (\| \Phi \| \|(B^1)^k x^1\| \cdots \|(D^m)^{-k} y^m\|)^{1/k}$$
$$\leqslant \frac{\lambda_2{}^1 \cdots \lambda_2{}^n}{\mu_1{}^1 \cdots \mu_1{}^m},$$
$$\overline{\lim_{k\to\infty}} \| E^{-k}\Phi(x^1,..., y^m)\|^{1/k} \leqslant \overline{\lim_{k\to\infty}} (\| \Phi \| \|(B^1)^{-k} x^1\| \cdots \|(D^m)^{k} y^m\|)^{1/k}$$
$$\leqslant \frac{\mu_2{}^1 \cdots \mu_2{}^m}{\lambda_1{}^1 \cdots \lambda_1{}^n},$$

that is

$$\Phi(x^1,..., y^m) \in Z^E\left(\left[\frac{\lambda_1{}^1 \cdots \lambda_1{}^n}{\mu_2{}^1 \cdots \mu_2{}^m}, \frac{\lambda_2{}^1 \cdots \lambda_2{}^n}{\mu_1{}^1 \cdots \mu_1{}^m}\right]\right).$$

Hence we have proved the implication (iii) $\Rightarrow$ (iv).

Finally suppose that (iv) is verified.

Let

$$0 < \lambda_1{}^i \leqslant \lambda_2{}^i < +\infty, \quad x^i \in (X^i)^{B^i}([\lambda_1{}^i, \lambda_2{}^i]), \quad 1 \leqslant i \leqslant n,$$
$$0 < \mu_1{}^j \leqslant \mu_2{}^j < +\infty, \quad y^j \in (Y^j)^{D^j}([\mu_1{}^j, \mu_2{}^j]), \quad 1 \leqslant j \leqslant m, \quad \varphi \in \mathscr{H}.$$

Then

$$(t_1,..., t_n, s_1,..., s_m) \mapsto \Phi(U^1_{t_1}x^1,..., U^n_{t_n}x^n, V^1_{s_1}y^1,..., V^m_{s_m}y^m)$$

is bounded and norm continuous. In particular, the function

$$(t_1,..., t_n, s_1,..., s_m, r) \mapsto \langle W_r\Phi(U^1_{t_1}x^1,..., U^n_{t_n}x^n, V^1_{s_1}y^1,..., V^m_{s_m}y^m), \varphi\rangle$$

is bounded and continuous.

Let now be $f \in L^1(\mathbb{R})$ such that $0 \notin \operatorname{supp} \hat{f}$; then there exist $\epsilon_1,..., \epsilon_n$, $\delta_1,..., \delta_m > 0$ such that

$$[-\epsilon_1 - \cdots - \epsilon_n - \delta_1 - \cdots - \delta_m, \epsilon_1 + \cdots + \epsilon_n + \delta_1 + \cdots + \delta_m] \cap \operatorname{supp} \hat{f} = \varnothing$$

Consider arbitrary functions $g_i \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{g}_i \subset [-\epsilon_i, \epsilon_i]$, $1 \leqslant i \leqslant n$, and $h_j \in L^1(\mathbb{R})$, $\operatorname{supp} \hat{h}_j \subset [-\delta_j, \delta_j]$, $1 \leqslant j \leqslant m$. Using the inversion formula for the Fourier transforms, it is easy to verify that g_i and h_j are equal almost everywhere with uniformly continuous bounded functions.

Let $p_1, \dots, p_n, q_1, \dots, q_m \in \mathbb{R}$ be arbitrary: then:

$$\mathscr{F}^1 = \int_{-\infty}^{+\infty} e^{ip_1t_1} g_1(t_1)\, U^1_{t_1} x^1\, dt_1 \in (X^1)^{B^1}([e^{-\varepsilon_1 - p_1}, e^{\varepsilon_1 - p_1}]),$$

$$\vdots$$

$$\mathscr{G}^m = \int_{-\infty}^{+\infty} e^{-iq_m s_m} \overline{h_m}(s_m)\, V^m_{s_m} y^m\, ds_m \in (Y^m)^{D^m}([e^{-\delta_m + q_m}, e^{\delta_m + q_m}]),$$

so

$$\mathscr{H} = \int_{\mathbb{R}^{n+m}} e^{ip_1t_1} \cdots e^{iq_m s_m} g_1(t_1) \cdots h_m(s_m)\, \Phi(U^1_{t_1}x^1, \dots, V^m_{s_m} y^m)\, dt_1 \cdots ds_m$$

$$\in Z^E([e^{-\varepsilon_1 - p_1 - \cdots - \delta_m - q_m}, e^{\varepsilon_1 - p_1 + \cdots + \delta_m - q_m}]).$$

It follows succesively

$$\mathscr{H} - \int_{-\infty}^{+\infty} e^{i(p_1 + \cdots - q_m)r} f(r)\, W_r \Big(\mathscr{H} - \int_{\mathbb{R}^{n+m}} e^{ip_1t_1} \cdots e^{iq_m s_m} g_1(t_1) \cdots h_m(s_m)$$

$$\times \Phi(U^1_{t_1}x^1, \dots, V^m_{s_m} y^m)\, dt_1 \cdots ds_m\Big) = 0,$$

$$\int_{\mathbb{R}^{n+m}} e^{i(p_1t_1 + \cdots + q_m s_m)} \Big(\int_{-\infty}^{+\infty} g_1(t_1 - r) \cdots h_m(s_m - r) f(r)$$

$$\times \langle W_r \Phi(U^1_{t_1 - r}x^1, \dots, V^m_{s_m - r} y^m), \varphi\rangle\, dr\Big)\, dt_1 \cdots ds_m = 0.$$

Consequently, the continuous Lebesgue integrable function

$$(t_1, \dots, s_m) \mapsto \int_{-\infty}^{+\infty} g_1(t_1 - r) \cdots h_m(s_m - r) f(r)$$

$$\times \langle W_r \Phi(U^1_{j_1 - r}x^1, \dots, V^m_{s_m - r} y^m), \varphi\rangle\, dr$$

vanishes identically. In particular,

$$\int_{-\infty}^{+\infty} g_1(-r) \cdots h_m(-r) f(r) \langle W_r \Phi(U^1_{-r}x^1, \dots, V^m_{-r} y^m), \varphi\rangle\, dr = 0.$$

Since g_1 can be replaced with any translate $g_1(t_1 + \cdot)$, it follows that for any $t_1 \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} g_1(t_1 - r)\, g_2(-r) \cdots h_m(-r) f(r) \langle W_r \Phi(U^1_{-r}x^1, \dots, V^m_{-r} y^m), \varphi\rangle\, dr = 0.$$

Thus, denoting

$$k_1(r) = g_2(-r) \cdots h_m(-r) f(r) \langle W_r \Phi(U_{-r}^1 x^1, \ldots, V_{-r}^m y^m), \varphi \rangle,$$

we have $g_1 * k_1 = 0$, that is $\hat{g}_1 \hat{k}_1 = 0$. Since $g_1 \in L^1(\mathbb{R})$, supp $\hat{g}_1 \subset [-\epsilon_1, \epsilon_1]$ is arbitrary, it follows that $\hat{k}_1$ vanishes on $[-\epsilon_1, \epsilon_1]$. In particular,

$$\int_{-\infty}^{+\infty} g_2(-r) \cdots h_m(-r) f(r) \langle W_r \Phi(U_{-r}^1 x^1, \ldots, V_{-r}^m y^m), \varphi \rangle \, dr = 0.$$

After an inductive reasoning, we obtain that

$$\int_{-\infty}^{+\infty} f(r) \langle W_r \Phi(U_{-r}^1 x^1, \ldots, V_{-r}^m y^m), \varphi \rangle \, dr = 0;$$

that is

$$\int_{-\infty}^{+\infty} f(-r) \langle W_{-r} \Phi(U_r^1 x^1, \ldots, V_r^m y^m), \varphi \rangle \, dr = 0.$$

Applying Shilov's theorem as in the proof of the Theorem 2.1, we conclude that the function

$$r \to \langle W_{-r} \Phi(U_r^1 x^1, \ldots, V_r^m y^m), \varphi \rangle$$

is constant. So, for each $r \in \mathbb{R}$,

$$\langle W_{-r} \Phi(U_r^1 x^1, \ldots, V_r^m y^m), \varphi \rangle = \langle \Phi(x^1, \ldots, y^m), \varphi \rangle.$$

Since $\varphi \in \mathscr{H}$ is arbitrary, for any $r \in \mathbb{R}$

$$\langle W_{-r} \Phi(U_r^1 x^1, \ldots, V_r^m y^m) = \Phi(x^1, \ldots, y^m),$$
$$\Phi(U_r^1 x^1, \ldots, V_r^m y^m) = W_r \Phi(x^1, \ldots, y^m).$$

Finally, using a density argument, it follows (i): so we have proved also the implication (iv) $\Rightarrow$ (i). Q.E.D.

The following common consequence of Theorems 2.1 and 2.2 is an improvement of the unicity result from [8], Theorem 4.4, which extends [3], Corollary 1 of Theorem 2.3, in the case of one-parameter groups.

2.3. COROLLARY. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ bounded $\mathscr{F}$-continuous one-parameter groups in $B_{\mathscr{F}}(X)$ and C and D their analytic generators. Then the following statements are equivalent*:

(i) $U_t = V_t, \quad t \in \mathbb{R}$;

(ii) $B \subset D$;

(iii) $X^B([\lambda_1, \lambda_2]) \subset X^D([\lambda_1, \lambda_2]), \quad 0 < \lambda_1 \leq \lambda_2 < +\infty.$

We remark that Theorem 2.2 is an extension of Theorem 1.6. By Theorem 2.2, if X is a complex Banach algebra, $\mathscr{F}$ is a Banach space in duality with X and $\{U_t\}$ is a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, then U_t are multiplicative if and only if for each $0 < \lambda_1 \leqslant \lambda_2 < +\infty$, $0 < \mu_1 \leqslant \mu_2 < +\infty$, we have

$$X^B([\lambda_1, \lambda_2])\, X^B([\mu_1, \mu_2]) \subset X^B([\lambda_1\mu_1, \lambda_2\mu_2]).$$

Analogously, if X is a complex Banach space with involution, $\mathscr{F}$ is a Banach space in duality with X and $\{U_t\}$ is a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ then U_t commute with the involution of X if and only if for any $0 < \lambda_1 \leqslant \lambda_2 < +\infty$

$$X^B([\lambda_1, \lambda_2])^* \subset X^B\left(\left[\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right]\right).$$

In the next section we shall be interested in $\{U_t\}$-invariant $\mathscr{F}$-continuous linear projections of X onto $X^B([1, 1])$, so we find it opportune to formulate here also the following common consequence of Theorems 2.1 and 2.2:

2.4. Corollary. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B its analytic generator, $0 < \lambda < +\infty$ and P and $\mathscr{F}$-continuous linear mapping of X into $X^B([\lambda, \lambda])$. Then the following statements are equivalent*:

(i) $PU_t = \lambda^{it}P$, $t \in \mathbb{R}$;

(ii) *for each* $x \in \mathscr{D}_B$ *we have* $PBx = \lambda Px$;

(iii) *for each* $0 < \lambda_1 \leqslant \lambda_2 < +\infty$ *such that* $\lambda \notin [\lambda_1, \lambda_2]$ *we have*

$$PX^B([\lambda_1, \lambda_2]) = \{0\}.$$

If $(Y, \mathscr{G})$ and $(Z, \mathscr{H})$ are dual pairs of Banach spaces, then we denote by $B_{\mathscr{G},\mathscr{H}}(Y, Z)$ the normed space of all linear mappings $Y \to Z$ which are continuous in the $\mathscr{G}$-topology on Y and the $\mathscr{H}$-topology on Z.

It is easy to see that $B_{\mathscr{G},\mathscr{H}}(Y, Z)$ is a Banach space.

2.5. Corollary. *Let $(X, \mathscr{F})$, $(Y, \mathscr{G})$, $(Z, \mathscr{H})$ be dual pairs of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B its analytic generator, $\{V_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{G}$-continuous one-parameter group in $B_{\mathscr{G}}(Y)$, D its analytic generator, $\{W_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{H}$-continuous one-parameter group in $B_{\mathscr{H}}(Z)$ and E its analytic generator.*

Consider a bounded linear mapping

$$\pi : X \to B_{\mathscr{G},\mathscr{H}}(Y, Z)$$

such that for each $y \in Y$ and $\varphi \in \mathscr{H}$ the linear functional $x \mapsto \langle \Pi(x) y, \varphi \rangle$ is $\mathscr{F}$-continuous. Then the following statements are equivalent:

(i) *for each $x \in X$ and $t \in \mathbb{R}$*

$$\pi(U_t x) = W_t \pi(x) V_{-t} ;$$

(ii) *for each $x \in \mathscr{D}_B$ and $y \in \mathscr{D}_{D^{-1}}$ we have $\Pi(x) D^{-1}y \in \mathscr{D}_E$ and*

$$\pi(Bx) y = E\pi(x) D^{-1}y;$$

(iii) *for each $0 < \lambda_1 \leqslant \lambda_2 < +\infty$ and $0 < \mu_1 \leqslant \mu_2 < +\infty$*

$$\pi(X^B([\lambda_1, \lambda_2])) Y^D([\mu_1, \mu_2]) \subset Z^B([\lambda_1\mu_1, \lambda_2\mu_2]).$$

Proof. Define the bounded bilinear mapping

$$\Phi : X \times Y \to Z$$

by

$$\Phi(x, y) = \pi(x) y.$$

Then, for fixed $y, x \mapsto \Phi(x, y)$ belongs to $B_{\mathscr{F},\mathscr{H}}(X, Z)$ and, for fixed $x, y \mapsto \Phi(x, y)$ belongs to $B_{\mathscr{G},\mathscr{H}}(Y, Z)$. Applying Theorem 2.2 to $\{U_t\}$, $\{V_t\}$, $\{W_t\}$, and Φ, the present equivalences result. Q.E.D.

If H is a Hilbert space and $\{u_t\}$ a weakly continuous one-parameter group of unitaries on H, then it is well known that $\{u_t\}$ is strongly continuous. We recall that the analytic generator b of $\{u_t\}$ is selfadjoint and positive and for each $t \in \mathbb{R}$, $u_t = b^{it}$ (see [8], Theorem 6.1).

2.6. COROLLARY. *Let X be a complex Banach space with involution, $\mathscr{F}$ a Banach space in duality with X, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous automorphisms of the linear space with involution X, B its analytic generator, H a Hilbert space, $\{u_t\}_{t\in\mathbb{R}}$ a strongly continuous one-parameter group of unitaries on H and b its analytic generator. Consider a bounded *-preserving linear mapping*

$$\pi : X \to B(H)$$

which is continuous with the $\mathscr{F}$-topology on X and the weak operatorial topology on $B(H)$. Then the following statements are equivalent:

(i) *for each $x \in X$ and $t \in \mathbb{R}$*

$$\pi(U_t x) = u_t \pi(x) u_{-t} ;$$

(ii) *for each $0 < \lambda, \mu < +\infty$*

$$\pi(X^B((0, \lambda])) H^b((0, \mu]) \subset H^b((0, \lambda\mu]).$$

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from Corollary 2.5 and Lemma 1.3.

Suppose that (ii) is verified and let be $0 < \lambda, \mu < +\infty$, $x \in X^B([\lambda, +\infty))$ and $\xi \in H^b([\mu, +\infty))$. By Theorem 1.6, $x^* \in X^B((0, 1/\lambda])$, so for each $\eta \in H^b((0, \nu])$, $0 < \nu < +\infty$

$$\pi(x)^* \eta = \pi(x^*)\eta \in H^b\left(\left(0, \frac{\nu}{\lambda}\right]\right).$$

It follows that for each $\eta \in H^b([\lambda\mu, +\infty))^\perp = \overline{\bigcup_{\nu<\lambda\mu} H^b((0, \nu])}$

$$\pi(x)^* \eta \in \overline{\bigcup_{\rho<\mu} H^b((0, \rho])} = H^b([\mu, +\infty))^\perp,$$

thus

$$(\pi(x)\, \xi \mid \eta) = (\xi \mid \pi(x)^* \eta) = 0.$$

Consequently

$$\pi(x)\, \xi \in H^b([\lambda\mu, +\infty)).$$

We conclude that for each $0 < \lambda_1 \leq \lambda_2 < +\infty$ and $0 < \mu_1 \leq \mu_2 < +\infty$

$$\begin{aligned} &\pi(X^B([\lambda_1, \lambda_2]))\, H^b([\mu_1, \mu_2]) \\ &\quad \subset (\pi(X^B((0, \lambda_2]))\, H^b((0, \mu_2])) \cap (\pi(X^B([\lambda_1, +\infty)))\, H^b([\mu_1, +\infty))) \\ &\quad \subset H^b((0, \lambda_2\mu_2]) \cap H^b([\lambda_1\mu_1, +\infty)) = H^b([\lambda_1\mu_1, \lambda_2\mu_2]). \end{aligned}$$

Using Corollary 2.5, it is clear that (i) results. Q.E.D.

Let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, $\{U_t\}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous $*$-automorphisms of X, and $\pi : X \to B(H)$ a $*$-representation of X which is continuous with the $\mathscr{F}$-topology on X and the weak operatorial topology on $B(H)$.

Suppose that there exists a family $\{H_\mu\}_{0\leq\mu<+\infty}$ of closed linear subspaces of H such that

$$H_\mu \subset H_\nu, \qquad \mu \leq \nu,$$

$$H_\mu = \bigcap_{\mu<\nu} H_\nu,$$

$$H_0 = \{0\},$$

$$\overline{\bigcup_\mu H_\mu} = H,$$

$$\pi(X^B((0, \lambda]))\, H_\mu \subset H_{\lambda\mu}.$$

Denoting by p_ν the ortogonal projection onto H_ν and $u_t = \int_0^{+\infty} \nu^{it} dp_\nu$, it follows that $\{u_t\}$ is a strongly continuous one-parameter group of unitaries on H and $H^b((0, \mu]) = H_\mu$. Applying Corollary 2.6, we find that

$$\pi(U_t x) = u_t \pi(x)\, u_{-t}\,.$$

This treatment of implementation problems goes back to the works of Helson and D. Lowdenslager [14], was very clearly formulated and developed by Forelli [13] for commutative C^*-algebras and was developed and very efficiently applied by Arveson [3] in the general case.

3. Ergodic Properties

The main purpose of this section is to establish a connection between the $\mathscr{F}$-convergence of the ergodic means $(1/2\epsilon)\, \mathscr{F}\text{-}\int_{-\epsilon}^{\epsilon} U_t x\, dt$ when $\epsilon \to +\infty$ and the behavior of B_ϵ when $\epsilon \to +\infty$ or $\epsilon \to -\infty$.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and $\{U_t\}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. For each $x \in X$ and $g \in L^1(\mathbb{R})$ such that

$$g(t) = g(-t), \qquad t \in \mathbb{R},$$

$$\int_{-\infty}^{+\infty} g(t)\, dt = 1,$$

we denote

$$\mathscr{B}^U_{\delta,\infty}(x, g) = \mathscr{F}\text{-closure of } \left\{ \mathscr{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon}\, g\left(\frac{t}{\epsilon}\right) U_t x\, dt;\ \epsilon \geq \delta \right\}, \quad \delta > 0,$$

$$\mathscr{B}_\infty{}^U(x, g) = \bigcap_{\delta>0} \mathscr{B}^U_{\delta,\infty}(x, g).$$

Since the mapping

$$(0, +\infty) \ni \epsilon \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon}\, g\left(\frac{t}{\epsilon}\right) U_t x\, dt$$

is norm-continuous, if $\mathscr{B}^U_{\delta,\infty}(x, g)$ is $\mathscr{F}$-compact for some δ, then it is $\mathscr{F}$-compact for each δ. Also, if the limit

$$\mathscr{F} - \lim_{\epsilon \to +\infty} \mathscr{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon}\, g\left(\frac{t}{\epsilon}\right) U_t x\, dt$$

exists, then $\mathscr{B}^U_{\delta,\infty}(x, g)$ are $\mathscr{F}$-compact.

We say that $\{U_t\}$ has the *weak ergodic property in* $x \in X$ if for each g as above, the sets $\mathscr{B}^U_{\delta,\infty}(x, g)$ are $\mathscr{F}$-compact; in this case $\mathscr{B}_\infty{}^U(x, g) \neq \varnothing$ for all g.

By the Alaoglu theorem, if $X = \mathcal{F}^*$ then each bounded $\mathcal{F}$-continuous one-parameter group in $B_{\mathcal{F}}(X)$ has the weak ergodic property in all $x \in X$.

3.1. THEOREM. *Let $(X, \mathcal{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathcal{F}$-continuous one-parameter group in $B_{\mathcal{F}}(X)$, and $g \in L^1(\mathbb{R})$ such that $g(t) = g(-t)$, $t \in \mathbb{R}$, $\hat{g} = 1$ on a neighborhood of* 0 *and* supp $\hat{g}$ *is compact.*

If $\{U_t\}_{t\in\mathbb{R}}$ has the weak ergodic property in $x \in X$ and $x_0 \in \mathcal{B}_\infty^U(x, g)$, then $x_0 \in X^B([1, 1])$ and $x - x_0$ belongs to the $\mathcal{F}$-closure of

$$\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)).$$

If $\{U_t\}_{t\in\mathbb{R}}$ has the weak ergodic property in $x \in X^B((0, 1])$ and $x_0 \in \mathcal{B}_\infty^U(x, g)$, then $x_0 \in X^B([1, 1])$ and $x - x_0$ belongs to the $\mathcal{F}$-closure of

$$\bigcup_{0<\mu<1} X^B((0, \mu]).$$

If $\{U_t\}_{t\in\mathbb{R}}$ has the weak ergodic property in $x \in X^B([1, +\infty))$ and $x_0 \in \mathcal{B}_\infty^U(x, g)$, then $x_0 \in X^B([1, 1])$ and $x - x_0$ belongs to the $\mathcal{F}$-closure of

$$\bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)).$$

Proof. Since supp $\hat{g}$ is compact, supp $\hat{g} \subset [-\theta, \theta]$ for some $\theta > 0$. Then for each $\epsilon > 0$

$$\mathcal{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right) U_t x \, dt \in X^B([e^{-(\theta/\epsilon)}, e^{\theta/\epsilon}])$$

and using [8], Lemma 5.5, it is easy to establish that

$$x - \mathcal{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right) U_t x \, dt \in \mathcal{F}\text{-closure of}$$
$$\left(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty))\right).$$

It follows that

$$\mathcal{B}_\infty^U(x, g) \subset \bigcap_{\epsilon>0} X^B([e^{-(\theta/\epsilon)}, e^{\theta/\epsilon}]) = X^B([1, 1])$$

and

$$x - \mathcal{B}_\infty^U(x, g) \subset \mathcal{F}\text{-closure of} \left(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty))\right).$$

If $x \in X^B((0, 1])$, then for each $\epsilon > 0$

$$x - \mathscr{F} - \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right) U_t x \, dt \in X^B((0, \mu_\epsilon]) \qquad \text{for some} \quad 0 < \mu_\epsilon < 1,$$

so

$$x - \mathscr{B}_\infty{}^U(x, g) \subset \mathscr{F}\text{-closure of} \bigcup_{0<\mu<1} X^B((0, \mu]).$$

Analogously, if $x \in X^B([1, +\infty))$ then

$$x - \mathscr{B}_\infty{}^U(x, g) \subset \mathscr{F}\text{-closure of} \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)).$$

Q.E.D.

3.2. Corollary. *Let $\mathscr{F}$ be a complex Banach space, $X = \mathscr{F}^*$, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(x)$, and $0 < \lambda < +\infty$. Then*

$$X = X^B([\lambda, \lambda]) + \mathscr{F}\textit{-closure of}\left(\bigcup_{0<\mu<\lambda} X^B((0, \mu]) + \bigcup_{\lambda<\mu<+\infty} X^B([\mu, +\infty))\right),$$

$$X^B((0, \lambda]) = X^B([\lambda, \lambda]) + \mathscr{F}\textit{-closure of} \bigcup_{0<\mu<\lambda} X^B((0, \mu]),$$

$$X^B([\lambda, +\infty)) = X^B([\lambda, \lambda]) + \mathscr{F}\textit{-closure of} \bigcup_{\lambda<\mu<+\infty} X^B([\mu, +\infty)).$$

Proof. The group $\{\lambda^{-it} U_t\}$ has the weak ergodic property in all $x \in X$ and applying Theorem 3.1 to this group, one can easily deduce the equalities given in the statement. Q.E.D.

We shall make use of the following consequence of Wiener's Tauberian theorem:

3.3. Lemma. *Let $f \in L^\infty(\mathbb{R})$ and $c \in \mathbb{C}$; then the following statements are equivalent*:

(i) $$\lim_{\epsilon \to +\infty} (1/2\epsilon) \int_{-\epsilon}^{\epsilon} f(t) \, dt = c;$$

(ii) $$\lim_{\epsilon \to +\infty} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\epsilon}{\epsilon^2 + t^2} f(t) \, dt = c;$$

(iii) *for each $g \in L^1(\mathbb{R})$ such that $g(t) = g(-t)$, $t \in \mathbb{R}$, we have*

$$\lim_{\epsilon \to +\infty} \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right) f(t) \, dt = c \int_{-\infty}^{+\infty} g(t) \, dt.$$

Proof. Suppose that there exists $g_0 \in L^1(\mathbb{R})$ such that $g_0(t) = g_0(-t)$, $t \in \mathbb{R}$, $\int_0^{+\infty} g_0(t)\, t^{is}\, dt \neq 0$ for all $s \in \mathbb{R}$ and

$$\lim_{\epsilon \to +\infty} \int_0^{+\infty} \frac{1}{\epsilon} g_0\left(\frac{t}{\epsilon}\right)(f(t) + f(-t))\, dt = \lim_{\epsilon \to +\infty} \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g_0\left(\frac{t}{\epsilon}\right) f(t)\, dt$$
$$= c \int_{-\infty}^{+\infty} g_0(t)\, dt = 2c \int_0^{+\infty} g_0(t)\, dt.$$

Applying a well-known form of Wiener's Tauberian theorem (see [12, Exercise XI 5.10] or [7, Exercise 2 from Section 3]), it follows that for each $g \in L^1(\mathbb{R})$ such that $g(t) = g(-t)$, $t \in \mathbb{R}$, we have

$$\lim_{\epsilon \to +\infty} \int_{-\infty}^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right) f(t)\, dt = \lim_{\epsilon \to +\infty} \int_0^{+\infty} \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right)(f(t) + f(-t))\, dt$$
$$= 2c \int_0^{+\infty} g(t)\, dt = c \int_{-\infty}^{+\infty} g(t)\, dt.$$

Now putting

$$g_0(t) = \tfrac{1}{2}\chi_{[-1,1]}(t),$$

where $\chi_{[-1,1]}$ is the characteristic function of $[-1, 1]$, we have $g_0(t) = g_0(-t)$, $t \in \mathbb{R}$, and for each $s \in \mathbb{R}$

$$\int_0^{+\infty} g_0(t)\, t^{is}\, dt = \frac{1}{2(1 + is)} \neq 0.$$

Using these facts, the implication (i) $\Rightarrow$ (ii) follows. The converse implication is trivial.

Finally let

$$g_0(t) = \frac{1}{\pi} \cdot \frac{1}{1 + t^2}.$$

Then $g_0(t) = g_0(-t)$, $t \in \mathbb{R}$, and for each $s \in \mathbb{R}$, applying [8], Corollary 3.5 for the case U_r = multiplication with $e^{i\pi s r}$, $r \in \mathbb{R}$, we obtain:

$$\int_0^{+\infty} g_0(t)\, t^{is}\, dt = \int_{-\infty}^{+\infty} \frac{e^{i\pi s r}}{e^{\pi r} + e^{-\pi r}}\, dr = \frac{1}{e^{\pi s/2} + e^{-\pi s/2}} \neq 0.$$

By the first part of the proof, it follows that (ii) $\Rightarrow$ (iii). The converse implication is again trivial. Q.E.D.

We say that $\{U_t\}$ has the *ergodic property in* $x \in X$ if the limit

$$\mathscr{F} - \lim_{\epsilon \to +\infty} (1/2\epsilon)\mathscr{F} - \int_{-\epsilon}^{\epsilon} U_t x\, dt$$

exists.

Denote this limit by $B_\infty x$. Following Lemma 3.3, if $\{U_t\}$ has the ergodic property in x, then it has also the weak ergodic property in x and for all $g \in L^1(\mathbb{R})$ such that

$$g(t) = g(-t), \qquad t \in \mathbb{R},$$

$$\int_{-\infty}^{+\infty} g(t)\, dt = 1,$$

we have

$$\mathscr{B}_\infty{}^U(x, g) = \{B_\infty x\}.$$

If $\{U_t\}$ satisfies the ergodic property in all $x \in X$, then

$$B_\infty : x \mapsto B_\infty x$$

is a bounded linear projection of X onto $X^B([1, 1])$ and

$$B_\infty U_t x = B_\infty x, \qquad t \in \mathbb{R}.$$

We say that $\{U_t\}$ has the *global ergodic property* if it has the ergodic property in all $x \in X$ and B_∞ is $\mathscr{F}$-continuous. If $\{U_t\}$ has the global ergodic property, then also $\{U_t{}^*\}$ has this property and for each $\varphi \in \mathscr{F}$ the limit

$$X - \lim_{\epsilon \to \infty} (1/2\epsilon) X - \int_{-\epsilon}^{\epsilon} U_t{}^* \varphi dt$$

is equal to the value $B_\infty{}^*\varphi$ of the adjoint of B_∞ in φ.

3.4. Theorem. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$; then the following statements are equivalent*:

(i) *$\{U_t\}_{t\in\mathbb{R}}$ has the global ergodic property*;

(ii) *$\{U_t\}_{t\in\mathbb{R}}$ has the ergodic property in all $x \in X$ and $\{U_t{}^*\}_{t\in\mathbb{R}}$ has the weak ergodic property in all $\varphi \in \mathscr{F}$*;

(iii) *there exists an $\mathscr{F}$-continuous linear projection P of X onto X^B ([1, 1]) such that*

$$PU_t = P, \qquad t \in \mathbb{R},$$

and $\{U_t\}_{t\in\mathbb{R}}$ has the weak ergodic property in all $x \in X$;

(iv) *there exists an $\mathscr{F}$-continuous linear projection Q of X onto $X^B([1, 1])$ such that*

$$\operatorname{Ker} Q = \mathscr{F}\text{-closure of}\Big(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty))\Big)$$

and $\{U_t{}^*\}_{t\in\mathbb{R}}$ has the weak ergodic property in all $\varphi \in \mathscr{F}$.

Proof. Obviously, (i) implies all the other statements. Suppose that (ii) is verified and let $\varphi \in \mathscr{F}$ and $\{x_\iota\}$ be a net in X which converges to 0 in the Mackey topology associated to the $\mathscr{F}$-topology. Denoting by $\chi_{[-1,1]}$ the characteristic function of $[-1, 1]$, we put $g_0 = \frac{1}{2}\chi_{[-1,1]}$. Since $\mathscr{B}^{U^*}_{1,\infty}(\varphi, g_0)$ is X-compact, its convex hull is relatively X-compact, so

$$\left\langle \frac{1}{2\epsilon}\mathscr{F} - \int_{-\epsilon}^{\epsilon} U_t x_\iota \, dt, \varphi \right\rangle = \left\langle x_\iota , \frac{1}{2\epsilon}\mathscr{F} - \int_{-\epsilon}^{\epsilon} U_t{}^* \varphi \, dt \right\rangle \to 0$$

uniformly for $\epsilon \geq 1$. It follows that

$$\langle B_\infty x_\iota , \varphi \rangle \to 0.$$

Consequently B_∞ is $\mathscr{F}$-continuous and we have proved that (ii) $\Rightarrow$ (i).

Now we shall suppose that (iii) is verified.

Let $x \in X$ be arbitrary. Suppose that the ergodic means $(1/2\epsilon)\mathscr{F} - \int_{-\epsilon}^{\epsilon} U_t x \, dt$ does not convergence to Px in the $\mathscr{F}$-topology when $\epsilon \to +\infty$; then there exist $\varphi \in \mathscr{F}$ and $\theta > 0$ such that for any $\delta > 0$

$$\left| \left\langle (1/2\epsilon)\mathscr{F} - \int_{-\epsilon}^{\epsilon} U_t x \, dt - Px, \varphi \right\rangle \right| \geq \theta \qquad \text{for some} \quad \epsilon \geq \delta.$$

So, for $g_0 = \frac{1}{2}\chi_{[-1,1]}$ and for any $\delta > 0$, the set

$$K_\delta = \{ y \in \mathscr{B}^U_{\delta,\infty}(x, g_0); |\langle y - Px, \varphi \rangle| \geq \theta \}$$

is not empty. Since K_δ are $\mathscr{F}$-compact, there exists

$$x_0 \in \bigcap_{\delta > 0} K_\delta .$$

It is easy to verify that for each $t \in \mathbb{R}$, $\delta > 0$, and $y \in \mathscr{B}^U_{\delta,\infty}(x, g_0)$

$$\| y - U_t y \| \leq (t/\delta) \sup_s \| U_s \| \, \| x \|.$$

It follows that $x_0 \in X^B([1, 1])$, hence $x_0 = Px_0$.

On the other hand, for each $\delta > 0$ and $y \in \mathscr{B}^U_{\delta,\infty}(x, g_0)$ we have $Py = Px$. In particular, $Px_0 = Px$.

It follows that

$$x_0 = Px$$

and this contradicts the fact that

$$|\langle x_0 - Px, \varphi \rangle| \geq \theta.$$

Consequently $\{U_t\}$ has the ergodic property in all $x \in X$ and $B_\infty = P$ is $\mathscr{F}$-continuous. So we have proved that (iii) $\Rightarrow$ (i).

Finally, suppose that (iv) is verified.

By Corollary 2.4, we have

$$QU_t = Q, \qquad t \in \mathbb{R}.$$

It is clear that Q is an X-continuous linear projection of $\mathscr{F}$ onto a subspace of $\mathscr{F}^{B^*}([1, 1])$ and

$$Q^*U_t^* = Q^*, \qquad t \in \mathbb{R}.$$

But if $\varphi \in \mathscr{F}^{B^*}([1, 1])$ and $x \in X$ are arbitrary, then

$$Qx - x \in \operatorname{Ker} Q \subset \mathscr{F}\text{-closure of} \left(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)) \right)$$

and using Corollary 1.5 we obtain

$$\langle x, Q^*\varphi - \varphi \rangle = \langle Qx - x, \varphi \rangle = 0.$$

It follows that Q^* is a projection onto $\mathscr{F}^{B^*}([1, 1])$.

Applying to $\{U_t^*\}$ the implication (iii) $\Rightarrow$ (i) proved above, we find that $\{U_t^*\}$ has the global ergodic property. Consequently $\{U_t\}$ has the global ergodic property too and the proof of (iv) $\Rightarrow$ (i) is finished. Q.E.D.

In the case $X = \mathscr{F}^*$ the statements of the above theorem can be improved:

3.5. COROLLARY. *Let $\mathscr{F}$ be a complex Banach space, $X = \mathscr{F}^*$, and $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. Then the following statements are equivalent*:

(i) $\{U_t\}_{t\in\mathbb{R}}$ *has the global ergodic property*;

(ii) *there exists an $\mathscr{F}$-continuous linear projection P of X onto $X^B([1, 1])$ such that*

$$PU_t = P, \qquad t \in \mathbb{R};$$

(iii) *for each $\varphi \in \mathscr{F}$*

$$\text{norm-}\lim_{\epsilon\to+\infty} (1/2\epsilon)X - \int_{-\epsilon}^{\epsilon} U_t^*\varphi \, dt$$

exists;

(iv) $$\bigcup_{0<\mu<1} \mathscr{F}^{B^*}((0, \mu]) + \mathscr{F}^{B^*}([1, 1]) + \bigcup_{1<\mu<+\infty} \mathscr{F}^{B^*}([\mu, +\infty))$$

is X-dense in $\mathscr{F}$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows immediately from Theorem 3.4. It is obvious that (iii) $\Rightarrow$ (i) and using Theorem 3.1, one can easily obtain that (i) $\mapsto$ (iv); thus it remains to prove only the implication (iv) $\Rightarrow$ (iii).

Suppose that (iv) is true. It is easy to see that

$$\mathscr{D} = \left\{\varphi \in \mathscr{F};\ \text{norm-lim}(1/2\epsilon)X - \int_{-\epsilon}^{\epsilon} U_t{}^*\varphi\, dt \text{ exists}\right\}$$

is a norm-closed linear subspace on $\mathscr{F}$, so it is X-closed.

Obviously, $\mathscr{F}^{B^*}([1, 1]) \subset \mathscr{D}$.

Let $\varphi \in \mathscr{F}^{B^*}((0, \mu])$, $0 < \mu < 1$; using the Cauchy integral theorem we have for each $\epsilon, \delta > 0$

$$\begin{aligned}(1/2\epsilon)X &- \int_{-\epsilon}^{\epsilon} U_t{}^*\varphi\, dt \\ &= (1/2\epsilon)X - \int_{-\epsilon}^{\epsilon} U_t{}^*B_\delta{}^*\varphi\, dt + (1/2\epsilon)X - \int_0^\delta (U_{-\epsilon}^*B_s{}^*\varphi - U_\epsilon{}^*B_s{}^*\varphi)\, ds,\end{aligned}$$

so that

$$\begin{aligned}&\left\| \frac{1}{2\epsilon} X - \int_{-\epsilon}^{\epsilon} U_t{}^*\varphi\, dt \right\| \\ &\qquad \leqslant \mu^\delta \sup_t \| U_t{}^* \|\, \| \varphi \| + (1/\epsilon) \sup_t \| U_t{}^* \| \int_0^\delta \mu^s\, ds\, \| \varphi \| .\end{aligned}$$

Letting $\delta \mapsto +\infty$, we we obtain for each $\epsilon > 0$

$$\left\| \frac{1}{2\epsilon} X - \int_{-\epsilon}^{\epsilon} U_t{}^*\varphi\, dt \right\| \leqslant \frac{1}{\epsilon} \sup_t \| U_t{}^* \| \left(\ln \frac{1}{\mu}\right)^{-1} \| \varphi \| .$$

Thus $x \in \mathscr{D}$.

Analogously, for each $1 < \mu < +\infty$ we have $\mathscr{F}^{B^*}([\mu, +\infty)) \subset \mathscr{D}$. Consequently $\mathscr{D}$ is X-dense in $\mathscr{F}$ and hence $\mathscr{D} = \mathscr{F}$. Q.E.D.

It is easy to formulate for the case $\mathscr{F} = X^*$ an analogous statement to Corollary 3.5.

In particular, if X is a reflexive Banach space and $\{U_t\}$ is a bounded strongly continuous one-parameter group in $B(X)$, then for each $x \in X$

$$\underset{\epsilon \to +\infty}{\text{norm-lim}}\ (1/2\epsilon) \int_{-\epsilon}^{\epsilon} U_t x\, dt$$

exists. We remark that this fact is also the consequence of a classical result (see [17, Satz 1.3.1]).

3.6. Theorem. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and $\{U_t\}_{t \in \mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. For each $x \in X^B((0, 1])$ and $x_0 \in X$ the following statements are equivalent*:

(i) *$\{U_t\}_{t \in \mathbb{R}}$ has the ergodic property in x and $B_\infty x = x_0$*;

(ii) *$\mathscr{F}$-$\lim_{\epsilon \to +\infty} B_\epsilon x = x_0$*.

For each $x \in X^B([1, +\infty))$ *and* $x_0 \in X$ *the following statements are equivalent*:

(j) $\{U_t\}_{t\in\mathbb{R}}$ *has the ergodic property in* x *and* $B_\infty x = x_0$;

(jj) $\mathscr{F}\text{-}\lim_{\epsilon\to\infty} B_\epsilon x = x_0$.

Proof. Following Lemma 1.2, for each $x \in X^B((0, 1])$ and $\epsilon > 0$

$$B_\epsilon x = \frac{1}{\pi}\, \mathscr{F} - \int_{-\infty}^{+\infty} \frac{\epsilon}{\epsilon^2 + t^2}\, U_t x\, dt.$$

Using Lemma 3.3, it is easy to conclude that (i) $\Leftrightarrow$ (ii).

The proof of the second equivalence is similar. Q.E.D.

Theorem 3.6 justifies the notation B_∞ for the limit of the ergodic means: Their limit appears as the "analytical extension of $\{U_t\}$ in ∞." By Theorem 3.6 we can expect that properties of the operators B_ϵ are preserved for the restrictions of B_∞ to $X^B((0, 1])$ and $X^B([1, +\infty))$.

We shall formulate now a result which tacitly was also supposed true above:

3.7. Corollary. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces and* $\{U_t\}_{t\in\mathbb{R}}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$ *which has the global ergodic property. Then*:

$$\operatorname{Ker} B_\infty = \mathscr{F}\text{-closure of} \left(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)) \right),$$

$$X^B((0, 1]) \cap \operatorname{Ker} B_\infty = \mathscr{F}\text{-closure of} \bigcup_{0<\mu<1} X^B((0, \mu]),$$

$$X^B([1, +\infty)) \cap \operatorname{Ker} B_\infty = \mathscr{F}\text{-closure of} \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)).$$

Proof. The inclusion

$$\operatorname{Ker} B_\infty \subset \mathscr{F}\text{-closure of} \left(\bigcup_{0<\mu<1} X^B((0, \mu]) + \bigcup_{1<\mu<+\infty} X^B([\mu, +\infty)) \right)$$

is a consequence of Theorem 3.1. To prove the converse inclusion we can use either Theorem 3.6 or Corollary 2.4 and Lemma 1.3.

The other equalities can be proved in the same manner. Q.E.D.

For each integer $n \geqslant 1$ we denote by $\mathrm{Mat}_n(\mathbb{C})$ the C^*-algebra of all $n \times n$ matrices over the complex field and by I_n the identity mapping $\mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$.

Let X, Y be C^*-algebras and $\Phi : X \to Y$ a linear mapping; Φ is called *completely positive* if for each integer $n \geq 1$, the mapping:

$$\Phi \otimes I_n : X \otimes \mathrm{Mat}_n(\mathbb{C}) \to Y \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive. If $\pi : Y \to B(H)$ is a faithful $*$-representation of Y, then Φ is completely positive if and only if for each $x_1, ..., x_n \in X$ and $\xi_1, ..., \xi_n \in H$, we have

$$\sum_{i,j=1}^{n} (\pi\Phi(x_i^* x_j)\, \xi_j \mid \xi_i) \geq 0.$$

For facts concerning completely positive mappings we go to [2]. We remark that the completely positive mappings are an important tool in the theory of C^*-algebras (see, for example, [4, 27]).

Now we shall again consider cases when X has additional structures compatible with the duality and $\{U_t\}$ preserves these structures.

3.8. THEOREM. *Let X be a complex Banach algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous automorphisms of the algebra X which has the global ergodic property. Then*:

(i) *for each $x \in X$ and $y_1, y_2 \in X^B([1, 1])$*

$$B_x(y_1 x y_2) = y_1 B_x(x)\, y_2\ ;$$

(ii) *B_x is multiplicative on $X^B((0, 1])$ and on $X^B([1, +\infty))$.*

If X is a C^*-algebra and U_t are *-automorphisms, then

(iii) *B_x is completely positive.*

If X is a W^*-algebra, $\mathscr{F}$ its predual and U_t are $*$-automorphisms, then

(iv) *B_x is faithful.*

Proof. The verification of (i) is trivial.

Let $x, y \in X^B((0, 1])$. By Corollary 3.7

$$x - B_x x \in X^B((0, 1]) \cap \mathrm{Ker}\, B_x = \mathscr{F}\text{-closure of} \bigcup_{0<\mu<1} X^B((0, \mu]).$$

Using Theorem 1.6, it is easy to find that

$$(x - B_x x)\, y \in \mathscr{F}\text{-closure of} \bigcup_{0<\mu<1} X^B((0, \mu]).$$

Again by Corollary 3.7

$$(x - B_\alpha x)\, y \in \operatorname{Ker} B_\alpha\,,$$

and using (i), we have

$$B_\alpha(xy) - (B_\alpha x)(B_\alpha y) = B_\alpha((x - B_\alpha x)\, y) = 0.$$

In the same way we can prove the multiplicativity of B_α on $X^B([1, +\infty))$.

Next suppose that X is a C^*-algebra and U_t are $*$-automorphisms. Let π be the direct sum of all cyclic $*$-representations of X associated to positive elements of $\mathscr{F}$. Then π is faithful and continuous with the $\mathscr{F}$-topology on X and the weak operatorial topology.

For each $x_1, ..., x_n \in X$ and for each vectors $\xi_1, ..., \xi_n$ in the representation space:

$$\sum_{i,j=1}^{n} (\pi B_\alpha(x_i^* x_j)\, \xi_j \mid \xi_i)$$

$$= \lim_{\epsilon \to +\infty} (1/2\epsilon) \int_{-\epsilon}^{\epsilon} \sum_{i,j=1}^{n} (\pi U_t(x_i^* x_j)\, \xi_j \mid \xi_i)\, dt$$

$$= \lim_{\epsilon \to +\infty} (1/2\epsilon) \int_{-\epsilon}^{\epsilon} \Big\| \sum_{j=1}^{n} \pi U_t(x_j)\, \xi_j \Big\|^2 dt \geq 0,$$

so B_α is completely positive.

Finally suppose that X is a W^*-algebra, $\mathscr{F}$ is its predual, and U_t are $*$-automorphisms.

It is clear that

$$\mathscr{I} = \{x \in X;\ B_\alpha(x^* x) = 0\}$$

is an $\mathscr{F}$-closed left ideal on X. By [24], Proposition 1.10.1 there exists an unique projection $e \in X$ such that $\mathscr{I} = Xe$. Since $\mathscr{I}$ is $\{U_t\}$-invariant, it follows that $e \in X^B([1, 1])$. Obviously $e \in \mathscr{I}$, so

$$e = B_\alpha e = 0,$$

$$\mathscr{I} = \{0\}.$$

Consequently B_α is faithful. Q.E.D.

We remark that statement (ii) of Theorem 3.8 can be extended, by replacing the product with a mapping like Φ in Theorem 2.2. The following completion of Corollary 3.5 was suggested to us by [18]; in fact, it is a consequence of the main result of Kovács and Szücs.

3.9. COROLLARY. *Let X be a W^*-algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X. Then the following statements are equivalent:*

(i) *$\{U_t\}_{t\in\mathbb{R}}$ has the global ergodic property;*

(ii) *if $x \in X^B([1, 1])$, $x \geq 0$, is such that for each $\varphi \in (X_*)^{B^*}([1, 1])$, $\varphi \geq 0$, we have $\langle x, \varphi\rangle = 0$, then $x = 0$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from Theorem 3.8(iv).

Conversely, suppose that (ii) is verified.

Let

$$x \in \overline{\left(\bigcup_{0<\mu<1} (X_*)^{B^*}((0, \mu]) + (X_*)^{B^*}([1, 1]) + \bigcup_{1<\mu<+\infty} (X_*)^{B^*}([\mu, +\infty))\right)}_X$$

and $\varphi \in (X_*)^{B^*}([1, 1])$ be arbitrary. By Corollary 1.5 $x \in X^B([1, 1])$, so we have succesively

$$x^* \in X^B([1, 1]),$$

$$L_{x^*}\varphi \in (X_*)^{B^*}([1, 1]),$$

$$\langle x^*x, \varphi\rangle = \langle x, L_{x^*}\varphi\rangle = 0.$$

Since $x^*x \in X^B([1, 1])$ and $x^*x \geq 0$, we conclude that $x^*x = 0$, that is $x = 0$. Using Corollary 3.5, it follows that $\{U_t\}$ has the global ergodic property. Q.E.D.

In the case of W^*-algebras and $*$-automorphisms we can characterize $X^B((0, 1]$ by the "vanishing of the negative Fourier coefficients":

3.10. COROLLARY. *Let X be a W^*-algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X which has the global ergodic property. Then*

$$X^B((0, 1]) = \{x \in X;\ B_\infty(yx) = 0 \text{ for all } y \in X^B((0, \mu]),\ 0 < \mu < 1\}$$
$$= \{x \in X:\ B_\infty(xy) = 0 \text{ for all } y \in X^B((0, \mu]),\ 0 < \mu < 1\}.$$

Proof. Denote

$$Y = \{x \in X;\ B_\infty(yx) = 0 \text{ for all } y \in X^B((0, \mu]),\ 0 < \mu < 1\}.$$

Then Y is an X_*-closed $\{U_t\}$-invariant linear subspace of X which contains $X^B((0, 1])$.

Let $1 < \lambda < +\infty$ and x be an element of the spectral subspace of $\{U_t \mid Y\}$ corresponding to $[\lambda, +\infty)$. Then $x \in X^B([\lambda, +\infty))$, so, by Theorem 1.6, $x^* \in X^B((0, 1/\lambda])$. Since $1/\lambda < 1$ and $x \in Y$, we have $B_\infty(x^*x) = 0$. Using Theorem 3.8(iv), it follows that $x = 0$.

Since $\{U_t \mid Y\}$ has the global ergodic property, by Theorem 3.1 Y must coincide with the spectral subspace of $\{U_t \mid Y\}$ corresponding to $(0, 1]$. In particular, $Y \subset X^B((0, 1])$.

In conclusion

$$Y = X^B((0, 1]).$$

The proof of the second equality is completely similar. Q.E.D.

Using Theorem 1.6(ii), it is easy to deduce similar characterizations for $X^B([1, +\infty))$.

In particular, $X^B((0, 1])$ and $X^B([1, +\infty))$ are maximal elements of the family of all subalgebras of X on which B_∞ is multiplicative.

Let X be a W^*-algebra and $\{U_t\}$ an X_*-continuous one-parameter group of $*$-automorphisms of X which has the global ergodic property. Denote $A = X^B((1, 1])$; then $A^* = X^B([1, +\infty))$ and $A \cap A^* = X^B([1, 1])$. By Theorem 3.8(iv), B_∞ is a faithful normal positive linear projection of X onto $A \cap A^*$, by Theorem 3.8(ii), B_∞ is multiplicative on A and by Theorem 3.1, $A + A^*$ is X_*-dense in X. Since $1 \in A \cap A^*$, A is a subdiagonal subalgebra of X with respect to B_∞ in the sense of [1], Definition 2.1.1. Moreover, by Corollary 3.10, A is a maximal subdiagonal subalgebra in the sense of [1], Definition 2.2.2. So the results of [1] are available.

In particular, let X be a finite W^*-algebra and $\{U_t\}$ an X_*-continuous one-parameter group of $*$-automorphisms of X such that for each nonzero positive $x \in X$ there exists an $\{U_t\}$-invariant normal finite trace φ with $\langle x, \varphi \rangle \neq 0$. By Corollary 3.9, $\{U_t\}$ has the global ergodic property. So, using [1], Theorem 4.2.1, for each inversible element $x \in X$ there exist

$$u \in X \qquad \text{unitary}$$

and

$$a \in X^B((0, 1]) \qquad \text{inversible with } a^{-1} \in X^B((0, 1])$$

such that

$$x = ua.$$

If $x = vb$ is another similar decomposition of x, then there exists a unitary $w \in X^B([1, 1])$ such that $v = uw$ and $b = w^*a$.

It would be interesting to decide if in the above situation the Jensen's inequality formulated in [1], Section 4.4, holds.

Finally let X be a W^*-algebra, ρ a faithful semifinite normal weight on X_+ and σ_t^ρ the associated modular $*$-automorphisms of X (see [9] or [26]). Then by [9], Theorem 3.4 and Corollary 3.5 the following statements are equivalent:

(i) $\{\sigma_t^\rho\}$ has the global ergodic property;

(ii) the restriction of ρ to $\{x \in X_+ ; \sigma_t^\rho(x) = x$ for all $t\}$ is semifinite;

(iii) there exists a family $\{\varphi_L\}$ of normal positive linear functionals on X whose supports are mutually orthogonal such that

$$\rho(x) = \sum_L \varphi_L(x), \qquad x \in X_+ .$$

In particular, if ρ is a faithful normal positive linear form, then $\{\sigma_t^\rho\}$ has the global ergodic property. We can ask also in this case for factorization results as in [1], Theorem 4.2.1.

4. Absolutely Continuous Elements

Denote $S^1 = \{\zeta \in \mathbb{C}; |\zeta| = 1\}$ and define the strongly continuous one-parameter group $\{U_t\}$ of *-automorphisms of the C^*-algebra $C(S^1)$ by

$$(U_t f)(\zeta) = f(e^{-it}\zeta), \qquad f \in C(S^1), \zeta \in S^1.$$

Considering on S^1 the normalized Lebesgue measure, $C(S^1)$ can be imbedded in a natural way in the W^*-algebra $L^\infty(S^1)$ and $\{U_t\}$ can be extended to an $L^1(S^1)$-continuous one-parameter group of $*$-automorphisms of $L^\infty(S^1)$. In this section we shall try to extend this type of imbedding to general one-parameter groups and to describe it more precisely in the case of C^*-algebras and $*$-automorphisms.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and $\{U_t\}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. We say that $\varphi \in \mathscr{F}$ is *absolutely continuous relative to* $\{U_t^*\}$ if $t \mapsto U_t^*\varphi$ is norm-continuous.

The set $\mathscr{F}^{B^*}$ of all absolutely continuous elements of $\mathscr{F}$ relative to $\{U_t^*\}$ is an $\{U_t^*\}$-invariant norm-closed linear subspace of $\mathscr{F}$. For each $0 < \lambda_1 < \lambda_2 < +\infty$ the spectral subspace of $\{U_t^* \mid \mathscr{F}^{B^*}\}$ associated to $[\lambda_1, \lambda_2]$ is $\mathscr{F}^{B^*}([\lambda_1, \lambda_2])$, so

$$\mathscr{F}^{B^*} = \text{norm-closure of} \bigcup_{0<\lambda_1 \leqslant \lambda_2 < +\infty} \mathscr{F}^{B^*}([\lambda_1, \lambda_2]).$$

Using [8], Lemma 5.5, it is easy to see that the closed unit ball of $\mathscr{F}$ is included in the X-closure of the closed ball with radius $8/\pi$ of $\mathscr{F}^{B^*}$.

For each $x \in X$, $\varphi \mapsto \langle x, \varphi\rangle$ is a linear bounded functional $\pi^B(x)$ on $\mathscr{F}^{B^*}$. So we have defined a linear mapping $\pi^B : X \to (\mathscr{F}^{B^*})^*$.

It is easy to verify that

$$(\pi/8)^{\prime\prime} \| x \| \leqslant \| \pi^B(x) \| \leqslant \| x \| , \qquad x \in X,$$

consequently π^B is an injective contraction and has norm-closed image.

Obviously, $\pi^B X$ is $\mathscr{F}^{B^*}$-dense in $(\mathscr{F}^{B^*})^*$.

Moreover, $\{(U_t^* \mid \mathscr{F}^{B^*})^*\}$ is a bounded $\mathscr{F}^{B^*}$-continuous one-parameter group in $B_{\mathscr{F}^{B^*}}((\mathscr{F}^{B^*})^*)$ and

$$\pi^B U_t = (U_t^* \mid \mathscr{F}^{B^*})^* \pi^B, \qquad t \in \mathbb{R}.$$

So, $\{U_t\}$ is "imbedded" via π^B in $\{(U_t^* \mid \mathscr{F}^{B^*})^*\}$. To describe this "imbedding" it suffices to locate $\mathscr{F}^{B^*}$ in $\mathscr{F}$.

We remark that if $X = \mathscr{F}^*$ then $\mathscr{F}^{B^*} = \mathscr{F}$, $(\mathscr{F}^{B^*})^* = X$ and the above "imbedding" is the identity.

Now let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous *-automorphisms of X. If $\varphi \in X^*$, $\varphi \geqslant 0$, then we denote by $\pi^\varphi : X \to B(H^\varphi)$ the associated cyclic *-representation and by ξ^φ the canonically associated cyclic vector. So

$$\langle x, \varphi \rangle = (\pi^\varphi(X)\, \xi^\varphi \mid \xi^\varphi), \qquad x \in X.$$

If $\varphi \in \mathscr{F}$, $\varphi \geqslant 0$, then π^φ is continuous with the $\mathscr{F}$-topology on X and the $B(H^\varphi)_*$-topology on $B(H^\varphi)$. We say that $\varphi \in \mathscr{F}$, $\varphi \geqslant 0$, is *quasi-invariant relative to* $\{U_t^*\}$ if there exists a strongly continuous one-parameter group $\{u_t^\varphi\}$ of unitaries on H^φ such that

$$\pi^\varphi(U_t x) = u_t^\varphi \pi^\varphi(x)\, u_{-t}^\varphi , \qquad x \in X.$$

It is easy to see that if $\varphi \in \mathscr{F}^{B^*}([1, 1])$, $\varphi \geqslant 0$, then φ is quasi-invariant relative to $\{U_t^*\}$: $\{u_t^\varphi\}$ can be defined by the equality

$$u_t^\varphi(\pi^\varphi(x)\, \xi^\varphi) = \pi^\varphi(U_t x)\, \xi^\varphi .$$

On the other hand, if $\varphi \in \mathscr{F}$, $\varphi \geqslant 0$, is quasi-invariant relative to $\{U_t^*\}$, then for each $x, y \in X^{**}$ the element $L_x R_y \varphi$ of $\mathscr{F}$ is absolutely continuous relative to $\{U_t^*\}$, that is it belongs to $\mathscr{F}^{B^*}$.

The following extension of a classical theorem of F. and M. Riesz (see [16, p. 47]) is due, under a norm-continuity hypothesis, to Arveson ([3], Theorem 5.3). Our proof is essentially the same as Arveson's proof.

4.1. THEOREM. *Let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, $\{U_t\}_{t\in\mathbb{R}}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous *-automorphisms of X, and $\varphi \in \bigcup_{0<\lambda<+\infty} \mathscr{F}^{B^*}((0, \lambda]) \cup \bigcup_{0<\lambda<+\infty} \mathscr{F}^{B^*}([\lambda, +\infty))$.*

Then $|\varphi|$ is quasi-invariant relative to $\{U_t^*\}_{t\in\mathbb{R}}$.

Proof. Suppose that $\varphi \in \mathscr{F}^{B*}((0, \lambda]), 0 < \lambda < +\infty$, and denote by $\pi = \pi_{\varphi}$, $H = H_{\varphi}$, $\xi = \xi_{|\varphi|}$. Let $v \in \pi(X)''$ be the partial isometry obtained via the polar decomposition of φ:

$$\langle x, \varphi \rangle = (\pi(x)\, v\xi \mid \xi), \qquad x \in X,$$
$$v^* v\xi = \xi.$$

For each $0 \le \mu < +\infty$ we define a closed linear subspace H_μ of H by

$$H_\mu = \bigcap_{\mu < \nu < +\infty} \overline{\pi(X^B((0, \nu]))\xi}.$$

Then

$$H_\mu \subset H_\nu, \qquad \mu \le \nu,$$
$$H_\mu = \bigcap_{\mu < \nu} H_\nu .$$

By Theorem 1.6(ii), for each $0 < \mu < +\infty$

$$\pi(X^B((0, \mu]))\, H_0 \subset H_0 .$$

Since $\bigcup_{0 < \mu < +\infty} \pi(X^B((0, \mu]))$ is $B(H)_*$-dense in $\pi(X)''$, it follows that

$$\pi(X)''\, H_0 \subset H_0 .$$

By Theorem 1.6(jj) and Corollary 1.5, for each $0 < \mu < 1/\lambda$ and $x \in X^B((0, \mu])$, we have

$$(v\xi \mid \pi(x)\, \xi) = (\pi(x^*)\, v\xi \mid \xi) = \langle x^*, \varphi \rangle = 0.$$

It follows that $v\xi$ is orthogonal to H_0. Since $\pi(X)''$ invariates H_0, also $\pi(X)''\, v\xi \supset \pi(X)''\, v^* v\xi = \pi(X)''\xi$ is orthogonal to H_0.

But as $\overline{\pi(X)''\xi} = H$, we have

$$H_0 = \{0\}.$$

Now, the $B(H)_*$-density of $\bigcup_{0<\mu<+\infty} \pi(X^B((0, \mu]))$ in $\pi(X)''$ and the cyclicity of ξ imply that

$$\overline{\bigcup_{\mu} H_\mu} = H.$$

Finally, by Theorem 1.6(ii), for each $0 < \rho, \mu < +\infty$

$$\pi(X^B((0, \rho]))\, H_\mu \subset H_{\rho\mu} .$$

Consequently, if p_ν is the orthogonal projection onto H_ν and $u_t = \int_0^{+\infty} \nu^{it}\, dp_\nu$, then $\{u_t\}$ is a strongly continuous one-parameter group of unitaries on H

whose spectral subspace corresponding to $(0,\mu]$, $0<\mu<+\infty$, is H_μ. Applying Corollary 2.6, we deduce that

$$\pi(U_t x) = u_t \pi(x) u_{-t}, \qquad x \in X,$$

so we have proved that $|\varphi|$ is quasi-invariant.

Similarly one can prove that if $\varphi \in \mathscr{F}^{B*}([\lambda, +\infty))$, $0<\lambda<+\infty$, then $|\varphi|$ is quasi-invariant. Q.E.D.

An interesting consequence of Theorem 4.1 and Lemma 1.3 is the following: If X is a C^*-algebra $\mathscr{F}$ is in duality with X and U_t are $\mathscr{F}$-continuous *-automorphisms, then for each $0<\lambda<+\infty$

$$\mathscr{F}^{B*}((0,\lambda]) = \text{norm-closure of } \bigcup_{0<\mu<\lambda} \mathscr{F}^{B*}([\mu,\lambda]),$$

$$\mathscr{F}^{B*}([\lambda,+\infty)) = \text{norm-closure of } \bigcup_{\lambda<\mu<+\infty} \mathscr{F}^{B*}([\lambda,\mu]).$$

By Theorem 4.1 we have the following description of $\mathscr{F}^{B*}$:

4.2. Corollary. *Let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}_{t\in\mathbb{R}}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous *-automorphisms of X. Then $\mathscr{F}^{B*}$ is the norm-closed translation-invariant linear hull of the set of all $\varphi \in \mathscr{F}$, $\varphi \geq 0$, φ quasi-invariant relative to $\{U_t{}^*\}_{t\in\mathbb{R}}$.*

Since $\mathscr{F}^{B*}$ is a norm-closed subspace of X^* and is invariant under translations with elements of X^{**}, using [24], Proposition 1.10.5, it follows that there exists a uniquely determined central projection p^B of X^{**} such that

$$\mathscr{F}^{B*} = L_{p^B} X^*,$$

$$(\mathscr{F}^{B*})^* = p^B X^{**},$$

$$\pi^B(x) = p^B x, \qquad x \in X.$$

In particular, $(\mathscr{F}^{B*})^*$ is a W^*-algebra and π^B is a *-homomorphism.

We remark that if $\{(U_t{}^* \mid \mathscr{F}^{B*})^*\}$ has the global ergodic property, then, using Theorem 3.8(iv), it results that $\mathscr{F}^{B*}$ is the norm-closed translation-invariant linear hull of the set of all $\varphi \in \mathscr{F}^{B*}([1,1])$, $\varphi \geq 0$.

For each $\varphi \in \mathscr{F}$ we define its *absolutely continuous part relative to* $\{U_t{}^*\}$ by

$$\varphi^{B*} = L_{p^B}\varphi.$$

We call $\varphi - \varphi^{B*}$ the *singular part* of φ *relative to* $\{U_t{}^*\}$.

We shall now give an extension of [16, 46, Corollary 1], generalizing in this way half of the classical Szegö–Kolmogorov–Krein theorem (see [16, p. 49]).

4.3 THEOREM. *Let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, $\{U_t\}_{t\in\mathbb{R}}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous $*$-automorphisms of X, $x \in X^B((0, 1])$, $0 < \lambda < 1$ and $\varphi \in \mathscr{F}$, $\varphi \geq 0$. Then*

$$\inf_{y\in X^B((0,\lambda])} \langle (x-y)^*(x-y), \varphi\rangle = \inf_{y\in X^B((0,\lambda])} \langle (x-y)^*(x-y), \varphi^{B^*}\rangle.$$

Proof. Denote by $\pi = \pi^\varphi$, $H = H^\varphi$, $\xi = \xi^\varphi$. Then π can be extended to a $*$-representation $X^{**} \to B(H)$ which is continuous with the weak toplologies defined by the preduals. We shall denote this extention by π also.

Let η be the orthogonal projection of $\pi(x)\,\xi$ on $\overline{\pi(X^B((0, \lambda]))\,\xi}$ and $\zeta = \pi(x)\,\xi - \eta$. Then

$$\inf_{y\in X^B((0,\lambda])} \langle (x-y)^*(x-y), \varphi\rangle = \|\zeta\|^2.$$

Let $\{y_n\}$ be a sequence in $X^B((0, \lambda])$ such that $\pi(y_n)\,\xi \to \eta$.

Since the positive linear functional φ_ζ defined on X by $\varphi_\zeta(z) = (\pi(z)\,\zeta \mid \zeta)$ is the limit in the norm topology of the functionals $L_{(x-y_n)^*} R_{x-y_n}\varphi \in \mathscr{F}$, it follows that $\varphi_\zeta \in \mathscr{F}$.

Let $y \in X^B((0, \lambda])$ be arbitrary; by Theorem 1.6(ii), $yx \in X^B((0, \lambda])$ so

$$(\pi(yx)\,\xi \mid \zeta) = 0.$$

On the other hand, again by Theorem 1.6(ii), for each n we have $yy_n \in X^B((0, \lambda])$, consequently

$$(\pi(y)\eta \mid \zeta) = \lim_n (\pi(yy_n)\xi \mid \zeta) = 0.$$

It follows that

$$\varphi_\zeta(y) = (\pi(y)\,\zeta \mid \zeta) = (\pi(yx)\,\xi \mid \zeta) - (\pi(y)\,\eta \mid \zeta) = 0.$$

We conclude that $\varphi_\zeta \in X^B((0, \lambda])^\perp_{\mathscr{F}}$. By Corollary 1.5, $\varphi_\zeta \in \mathscr{F}^{B^*}([\lambda, +\infty))$. Since φ_ζ is positive, by Theorem 1.6(ii), we have $\varphi_\zeta = \varphi_\zeta{}^* \in \mathscr{F}^{B^*}((0, 1/\lambda])$ and consequently

$$\varphi_\zeta \in \mathscr{F}^{B^*}([\lambda, 1/\lambda]) \subset \mathscr{F}^{B^*},$$

so

$$\varphi_\zeta = L_{p^B}\varphi_\zeta\,.$$

In particular,

$$\|\zeta\|^2 = \|\varphi_\zeta\| = \|L_{p^B}\varphi_\zeta\| = \|\pi(p^B)\zeta\|^2.$$

Since

$$\|(1 - \pi(p^B))\zeta\|^2 = \|\zeta\|^2 - \|\pi(p^B)\zeta\|^2 = 0,$$
$$(1 - \pi(p^B))\zeta = 0,$$

for each $y \in X^B((0, \lambda])$ we have

$$(\pi(p^B y)\xi \mid \pi(p^B)\zeta) = (\pi(y)\xi \mid \zeta) = 0.$$

It follows that $\pi(p^B)\eta$ is the orthogonal projection of $\pi(p^B x)\xi$ on $\overline{\pi(p^B X^B((0, \lambda]))\xi}$, consequently

$$\inf_{y \in X^B((0,\lambda])} \langle (x - y)^*(x - y), \varphi^{B^*}\rangle = \|\Pi(p^B)\zeta\|^2 = \|\zeta\|^2.$$

Q.E.D.

In particular, if $\varphi \in \mathscr{F}$, $\varphi \geq 0$, and $\varphi^{B^*} = 0$, then for each $0 < \lambda < +\infty$

$$\overline{\pi^{\varphi}(X^B((0, \lambda]))\xi^{\varphi}} = \overline{\pi^{\varphi}(X^B((0, 1]))\xi^{\varphi}}.$$

We remark that if $\varphi \in \mathscr{F}^{B^*}([1, 1])$, $\varphi \geq 0$, then for each $0 < \lambda < \mu < +\infty$

$$\pi^{\varphi}(X^B((0, \lambda]))\xi^{\varphi} \quad \text{is orthogonal to} \quad \pi^{\varphi}(X^B([\mu, +\infty)))\xi^{\varphi}.$$

Now let X be a von Neumann algebra, that is a $*$-subalgebra of some $B(H)$ such that $X'' = X$. Consider an X_*-continuous one-parameter group $\{U_t\}$ of $*$-automorphisms of X and suppose that $\{U_t\}$ has the global ergodic property. It is easy to see that for each $\varphi \in X_*$, $\varphi \geq 0$,

$$s(\varphi) \leq s(B_\infty{}^*\varphi).$$

Thus there exists a positive selfadjoint linear operator A in H, affiliated to X, such that

$$s(A) \leq s(B_\infty{}^*\varphi),$$
$$\varphi = L_A R_A(B_\infty{}^*\varphi)$$

in an appropriate sense, and if the projection $s(B_\infty{}^*\varphi)$ is finite, then A is uniquely determined by the above relations (see, for example, [25], Theorem 10.10). A possible extension of the second half of the Szegö–Kolmogorov–Krein theorem, that is of the Szegö theorem, would be related with the following problem: Find a formula for

$$\inf_{y \in X^B((0,\lambda])} \langle (x - y)^*(x - y), \varphi\rangle$$

in terms of $x \in X^B((0, 1])$, $\lambda \in (0, 1)$, $B_\infty{}^*\varphi$, and A. For a plausible formulation of an extended Szegö theorem in the case when $B_\infty{}^*\varphi$ is a trace and for various comments we send you to [1].

Finally, let X be a C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous $*$-automorphisms of X. Consider a positive element φ of $\mathscr{F}$ which is quasi-invariant relative to $\{U_t{}^*\}$ and let $\{u_t{}^\varphi\}$ be a strongly continuous one-parameter group of unitaries on H^φ such that

$$\pi^\varphi(U_t x) = u_t^\varphi \pi^\varphi(x) u_{-t}^\varphi.$$

Then, for each $t \in \mathbb{R}$,

$$\overline{\pi^\varphi(X)'' u_{-t}^\varphi \xi^\varphi} = \overline{u_{-t}^\varphi \pi^\varphi(X)'' \xi^\varphi} = H^\varphi = \overline{\pi^\varphi(X)'' \xi^\varphi},$$

so, by [24], Corollary 2.7.10, the orthogonal projection on $\overline{\pi^\varphi(X) u_{-t} \xi^\varphi}$ is equivalent in $\pi^\varphi(X)''$ to the orthogonal projection on $\overline{\pi^\varphi(X)' \xi^\varphi}$. In other words, for each $t \in \mathbb{R}$, the supports of $U_t{}^*\varphi$ and φ in X^{**} are equivalent. We also remark that if X is commutative, then this fact characterizes the quasi-invariantness of φ (see [13], Theorem 2).

Let be $X = C(S^1)$, $\mathscr{F} = X^* =$ the space $M(S^1)$ of all finite regular Borel measures on S^1 and

$$(U_t f)(\zeta) = f(e^{-it}\zeta).$$

If $\mu \in M(S^1)$, $\mu \geq 0$, is quasi-invariant relative to $\{U_t{}^*\}$ then for each $t \in \mathbb{R}$ the measures $U_t{}^*\mu$ and μ are equivalent in the sense of absolute continuity: conseqeuntly μ is either 0 or equivalent to the Lebesgue measure (see [6], Section 1, Proposition 11). It follows that $\mathscr{F}^{B^*}$ can be identified with $L^1(S^1)$.

On the other hand, $X^B((0, 1]) = \{f \in C(S^1);\, f$ has a regular extension on $|\zeta| \leq 1\}$ is the norm-closed linear hull of all $\zeta \to \zeta^n$, $n \geqslant 0$, so the norm-closure of $\bigcup_{0<\lambda<1} X^B((0, \lambda])$ is the norm-closed linear hull of all $\zeta \to \zeta^n$, $n \geqslant 1$. But by Corollary 1.5,

$$\mathscr{F}^{B^*}([1, +\infty)) = \left\{\mu \in M(S^1);\, \int_{S^1} \zeta^n \, d\mu(\zeta) = 0 \text{ for all } n \leqslant -1\right\},$$

and so, now using Theorem 4.1, the F. and M. Riesz theorem results.

Similar considerations can be made in the case $X = C_0(\mathbb{R})$, $\mathscr{F} = M(\mathbb{R})$ and

$$(U_t f)(s) = f(s - t).$$

5. Invariant Subspaces

Let χ be a function defined on S^1 by $\chi(\zeta) = \zeta$. A closed linear subspace K of $L^2(S^1)$ is called invariant if $\chi K \subset K$. By a theorem of N. Wiener (see [14, Theorem 2]), if K is doubly invariant, that is if $\chi K = K$, then for some Borel set $E \subset S^1$

$$K = \{f \in L^2(S^1);\, f \text{ vanishes on } E\}$$

and by a theorem of A. Beurling, H. Helson, and D. Lowdenslager (see [14, Theorem 3]), if K is simply invariant, that is if $XK \subset K$, $XK \neq K$, then for some $q \in L^\infty(S^1)$, $|q| = 1$,

$$K = qH^2(S^1).$$

In this section we try to extend these facts to the case when $X \subset B(H)$ is a von Neumann algebra, $\{U_t\}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, and $K \subset H$ is invariant under the action of elements of $X^B((0, 1])$.

Let H be a Hilbert space, $S \subset B(H)$, and $K \subset H$; then let us denote by $[SK]$ the closed linear hull of SK.

Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}$ an X_*-continuous one-parameter group of $*$-automorphisms of X. A closed linear subspace K of H is called *invariant relative to* $\{U_t\}$ if

$$\bigcap_{1<\lambda<+\infty} [X^B((0, \lambda])K] = K.$$

We say that an invariant subspace K is *doubly invariant relative to* $\{U_t\}$ if

$$\bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])K] = K$$

and we say that it is *simply invariant relative to* $\{U_t\}$ *if*

$$\bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])K] = \{0\}.$$

Let K be an invariant subspace relative to $\{U_t\}$; for $0 \leqslant \lambda \leqslant +\infty$, we shall denote by

$$K_\lambda = \bigcap_{\lambda<\mu<+\infty} [X^B((0, \mu])K] \qquad \text{if} \quad \lambda < \infty,$$

$$K_\infty = \overline{\bigcup_{0<\mu<+\infty} [X^B((0, \mu])K]}.$$

Obviously,

$$K_\lambda \subset K_\mu, \qquad \lambda \leqslant \mu,$$

$$K_\lambda = \bigcap_{\lambda<\mu} K_\mu,$$

$$K_1 = K$$

$$\overline{\bigcup_{\lambda<\infty} K_\lambda} = K_\infty.$$

Using Theorem 1.6(ii) and the X_*-density of $\bigcup_\mu X^B((0, \mu])$ in X, we obtain:

$$XK_0 \subset K_0 ,$$

$$K_\infty = [XK],$$

$$X^B((0, \lambda])\, K_\mu \subset K_{\lambda\mu} .$$

Denoting by p_λ^K the orthogonal projection on K_λ, it follows that p_0^K, $p_\infty^K \in X'$.

We call p_∞^K the *support of* K.

The following decomposition theorem of Wold type (compare with [22], Chap. I, Theorem 3.2) reduces the description of the invariant subspaces to the description of the doubly invariant and simply invariant subspaces.

5.1. THEOREM. *Let $X \subset B(H)$ be a von Neumann algebra, $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, and K an invariant subspace relative to $\{U_t\}_{t\in\mathbb{R}}$. Then there exists a unique projection $e' \in X'$ such that $e'H \subset K$, $e'H$ is doubly invariant relative to $\{U_t\}_{t\in\mathbb{R}}$ and $(1 - e')\,K$ is simply invariant relative to $\{U_t\}_{t\in\mathbb{R}}$.*

Moreover, $e' = p_0^K$ and the support of $(1 - e')\,K$ is $p_\infty^K - p_0^K$.

Proof. Obviously, $p_0^K H = K_0 \subset K$.

Since $XK_0 \subset K_0$, K_0 is invariant and since we have

$$\begin{aligned} K_0 = p_0^K \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])K] \\ \subset \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])\, p_0^K K] \\ = \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])\, K_0], \end{aligned}$$

it follows that K_0 is doubly invariant.

Analogously, since we have

$$\begin{aligned} &\bigcap_{1<\lambda<+\infty} [X^B((0, \lambda])(1 - p_0^K)K] \\ &\subset ((1 - p_0^K)H) \cap \bigcap_{1<\lambda<+\infty} [X^B((0, \lambda])K] \\ &= ((1 - p_0^K)H) \cap K \\ &= (1 - p_0^K)K, \end{aligned}$$

it follows that $(1 - p_0^K) K$ is invariant and finally, since

$$\bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])(1 - p^K)K] \subset ((1 - p_0^K)H) \cap \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])K] = ((1 - p_0^K)H) \cap (p_0^K H) = \{0\},$$

it results that $(1 - p_0^K) K$ is simply invariant.

It is easy to verify that the support of $(1 - p_0^K) K = (p_\infty^K - p_0^K) K$ is $p_\infty^K - p_0^K$.

Now let $e' \in X'$ be a projection such that $e'H \subset K$, $e'H$ is doubly invariant and $(1 - e') K$ is simply invariant. As we have

$$(1 - e') p_0^K H = (1 - e') \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])K] \subset \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])(1 - e')K] = \{0\},$$

it holds:

$$(1 - e') p_0^K = 0, \qquad p_0^K = e'p_0^K \leqslant e'.$$

On the other hand, since

$$(1 - p_0^K) e'H = (1 - p_0^K) \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda]) e'H] \subset \bigcap_{0<\lambda<+\infty} [X^B((0, \lambda])(1 - p_0^K)K] = \{0\},$$

we have also

$$(1 - p_0^K) e' = 0,$$

$$e' = p_0^K e' \leqslant p_0^K. \qquad \text{Q.E.D.}$$

A trivial invariant space is H and for each $0 \leqslant \lambda \leqslant \infty$ we have $X'H_\lambda \subset H_\lambda$; so $p_\lambda^H \in X$. In particular, p_0^H is a central projection in X.

Obviously, $p_\lambda^H = 1$ for $\lambda \geqslant 1$.

We still remark that by Theorem 5.1, H_0 is doubly invariant; moreover, each doubly invariant subspace is included in H_0.

The following result is an extension of Wiener's theorem:

5.2. THEOREM. *Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of *-automorphisms of X. If K is a doubly invariant subspace relative to $\{U_t\}_{t\in\mathbb{R}}$ then for each $0 < \lambda < +\infty$*

$$[X^B((0, \lambda]) K] = K,$$

so $p_\infty^K = p_0^K \leqslant p_0^H$ and $K = p_\infty^K H$.

Conversely, if $e' \in X'$ is a projection such that $e' \leq p_0^H$, then $e'H$ is doubly invariant relative to $\{U_t\}_{t\in\mathbb{R}}$.

Proof. Let K be doubly invariant and $0 < \lambda < +\infty$: then $K = K_0$, so we have:

$$K = K_0 \subset [X^B((0, \lambda])\, K] \subset [XK_0] \subset K_0 = K.$$

Conversely, let $e' \in X'$ be a projection such that $e' \leq p_0^H$. Since H_0 is doubly invariant, for each $0 < \lambda < +\infty$, we have

$$H_0 = [X^B((0, \lambda])\, H_0],$$

and this implies

$$[X^B((0, \lambda])\, e'H] \subset e'H = e'H_0 = e'[X^B((0, \lambda])\, H_0] \subset [X^B((0, \lambda])\, e'H].$$

It follows that $e'H$ is doubly invariant. Q.E.D.

Let $e' \in X'$ be a projection. We say that a family $\{u_t\}_{t\in\mathbb{R}} \subset B(H)$ is an implementing group with support e' for $\{U_t\}$ if:

$$u_t = u_t e' = e' u_t, \qquad t \in \mathbb{R},$$

$\{u_t \mid e'H\}$ is a strongly continuous group of unitaries on $e'H$,

$$U_t(x)\, e' = u_t x u_{-t}, \qquad x \in X,\ t \in \mathbb{R}.$$

We shall prove now a partial extension of the Beurling–Helson–Lowdenslager theorem, establishing a connection between simply invariant subspaces and implementing groups. The idea of this extension comes from an examination of proofs from [3].

5.3. THEOREM. *Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X.*

For each simply invariant supspace K relative to $\{U_t\}_{t\in\mathbb{R}}$, there exists an implementing group $\{u_t\}_{t\in\mathbb{R}}$ with support p_∞^K for $\{U_t\}_{t\in\mathbb{R}}$ such that for each $0 < \lambda < +\infty$ the spectral subspace of $\{u_t \mid p_\infty^K H\}_{t\in\mathbb{R}}$ corresponding to $(0, \lambda]$ is K_λ. Conversely, if $\{v_t\}_{t\in\mathbb{R}}$ is an implementing group with support e' for $\{U_t\}_{t\in\mathbb{R}}$ and $0 < \lambda_0 < +\infty$ then the spectral subspace L of $\{v_t \mid e'H\}_{t\in\mathbb{R}}$ corresponding to $(0, \lambda_0]$ is a simply invariant subspace relative to $\{U_t\}_{t\in\mathbb{R}}$, for each $0 < \lambda < +\infty$ L_λ is included in the spectral subspace of $\{v_t \mid e'H\}_{t\in\mathbb{R}}$ corresponding to $(0, \lambda_0\lambda]$, $p_\infty^L \leq e'$ and $v_t p_\infty^L = p_\infty^L v_t$, $t \in \mathbb{R}$.

Proof. Let K be a simply invariant subspace. Consider the $*$-representation $\pi : X \to B(p_\infty^K H)$ defined by

$$\pi(x) = x \mid p_\infty^K H.$$

Denote $q_\mu = p_\mu{}^K \mid p_\infty{}^K H$ and $w_t = \int_0^\infty \mu^{it}\, dq_\mu$. Then the spectral subspace of $\{w_t\}$ corresponding to $(0, \lambda]$, $0 < \lambda < +\infty$, is K_λ. Applying Corollary 2.6, it follows that

$$\pi(U_t x) = w_t \pi(x)\, w_{-t}, \qquad x \in X,\ t \in \mathbb{R}.$$

Putting

$$\begin{aligned} u_t\xi &= w_t\xi, && \xi \in p_\infty{}^K H, \\ &= 0, && \xi \in (1 - p_\infty{}^K)\, H, \end{aligned}$$

$\{u_t\}$ is an implementing group with support $p_\infty{}^K$ for $\{U_t\}$.

Conversely, let $\{v_t\}$ be an arbitrary implementing group for $\{U_t\}$, e' its support, $0 < \lambda_0 < +\infty$ and L the spectral subspace of $\{v_t \mid e'H\}$ corresponding to $(0, \lambda_0]$. Applying again Corollary 2.6, it follows that, for each $0 < \mu < +\infty$, $[X^B((0, \mu])\, L]$ is included in the spectral subspace of $\{v_t \mid e'H\}$ corresponding to $(0, \lambda_0\mu]$. It is easy to see now that L is simply invariant and that, for each $0 < \lambda < +\infty$, L_λ is included in the spectral subspace of $\{v_t \mid e'H\}$ corresponding to $(0, \lambda_0\lambda]$. In particular, $p_\infty{}^L \leqslant e'$. For each $t \in \mathbb{R}$ and $0 < \lambda < +\infty$

$$U_t X^B((0, \lambda]) = X^B((0, \lambda]), \qquad v_t L = L,$$

so

$$v_t[X^B((0, \lambda])\, L] = [(U_t X^B((0, \lambda]))\, v_t L] = [X^B((0, \lambda])\, L].$$

It follows that

$$v_t L_\infty \subset L_\infty, \qquad t \in \mathbb{R},$$

and consequently

$$v_t p_\infty{}^L = p_\infty{}^L v_t, \qquad t \in \mathbb{R}.$$

Q.E.D.

5.4. Corollary. *Let $X \subset B(H)$ be a von Neumann algebra, $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, and K an invariant subspace relative to $\{U_t\}_{t\in\mathbb{R}}$. If $p_\infty{}^K - p_0{}^K \in X$, then it belongs simultaneously to the center of X and to $X^B([1, 1])$.*

Proof. Suppose that $p_\infty{}^K - p_0{}^K \in X$. Since $p_\infty{}^K - p_0{}^K$ belongs to X', it is in the center of X. On the other hand, by Theorem 5.1, $(1 - p_0{}^K)\, K$ is simply invariant and its support is $p_\infty{}^K - p_0{}^K$. Applying Theorem 5.3 there exists an implementing group $\{u_t\}$ with support $p_\infty{}^K - p_0{}^K$ for $\{U_t\}$. In particular, for each $t \in \mathbb{R}$

$$U_t((p_\infty{}^K - p_0{}^K))(p_\infty{}^K - p_0{}^K) = u_t(p_\infty{}^K - p_0{}^K)\, u_{-t} = p_\infty{}^K - p_0{}^K,$$

$$p_\infty{}^K - p_0{}^K \leqslant U_t(p_\infty{}^K - p_0{}^K).$$

It follows that

$$U_t(p_\infty^K - p_0^K) = p_\infty^K - p_0^K, \qquad t \in \mathbb{R},$$

that is

$$p_\infty^K - p_0^K \in X^B([1, 1]).$$

Q.E.D.

In particular, $1 - p_0^H = p_\infty^H - p_0^H \in X^B([1, 1])$, so $p_0^H \in X^B([1, 1])$.

We say that an implementing group $\{u_t\}$ with support e' for $\{U_t\}$ is *minimal*, if, denoting by K the spectral subspace of $\{u_t \mid e'H\}$ corresponding to $(0, 1]$, the spectral subspace of $\{u_t \mid e'H\}$ corresponding to any $(0, \lambda]$ is K_λ. By Theorem 5.3 there exists a correspondence between simply invariant subspaces with support e' and minimal implementing groups with support e'.

5.5. Corollary. *Let $X \subset B(H)$ be a von Neumann algebra, $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, $\{v_t\}_{t\in\mathbb{R}}$ an implementing group with support e' for $\{U_t\}_{t\in\mathbb{R}}$, and p_λ the orthogonal projection of H on the spectral subspace of $\{v_t \mid e'H\}_{t\in\mathbb{R}}$ corresponding to $(0, \lambda]$, $0 < \lambda < +\infty$. Then for any sequence $0 < \lambda_1 < \lambda_2 < \cdots, \lambda_m \to +\infty$, there exists a sequence $e_1', e_2', \ldots \in X'$, $e_n'e_m' = 0$ for all $n \neq m$, $\sum_{n=1}^\infty e_n' = e'$, $v_te_n' = e_n'v_t$ for all n and t, such that the range of the projection $\sum_{n=1}^\infty p_{\lambda_n}e_n'$ is a simply invariant subspace with support e' relative to $\{U_t\}_{t\in\mathbb{R}}$.*

Proof. By Theorem 5.3, for any $0 < \lambda < +\infty$

$$p_\lambda H \text{ is simply invariant,}$$

$$X^B((0, \mu])\, p_\lambda H \subset p_{\mu\lambda}H, \qquad 0 < \mu < +\infty,$$

and, denoting by f_λ' the support of $p_\lambda H$,

$$v_tf_\lambda' = f_\lambda'v_t, \qquad t \in \mathbb{R}.$$

Denote by

$$e_1' = f'_{\lambda_1},$$

$$e_n' = f'_{\lambda_n} - f'_{\lambda_{n-1}}, \qquad n \geq 2;$$

then we have

$$e_n'e_m' = 0, \qquad n \neq m,$$

$$\sum_{n=1}^\infty e_n' = e',$$

$$v_te_n' = e_n'v_t \qquad \text{for all } n \text{ and } t.$$

Denote by $K = (\sum_{n=1}^{\infty} p_{\lambda_n} e_n') H$. If

$$\xi \in \bigcap_{1<\mu<+\infty} [X^B((0, \mu])K]$$

then, for any $n \geqslant 1$, we have:

$$e_n'\xi \in \bigcap_{1<\mu<+\infty} [X^B((0, \mu]) p_{\lambda_n} e_n' H] \subset \bigcap_{1<\mu<+\infty} p_{\mu\lambda_n} H = p_{\lambda_n} H,$$

and therefore

$$\xi = \sum_{n=1}^{\infty} e_n'\xi = \sum_{n=1}^{\infty} p_{\lambda_n} e_n' \xi \in K.$$

Hence K is invariant. A similar argument shows that it is simply invariant. Obviously, $p_\infty{}^K \leqslant e'$; on the other hand, it is easy to verify that $e' - p_\infty{}^K$ is orthogonal successively to $p_{\lambda_1}, f_1', p_{\lambda_2}, f_2', \ldots$, so it is orthogonal to e'. Consequently, $p_\infty{}^K = e'$. Q.E.D.

By Corollary 5.5, if there exists an implementing group with support e' then there exists a simply invariant subspace with support e' and so, using Theorem 5.3, there exists a minimal implementing group with support e'. Consequently, we have a procedure to improve implementing groups. The idea of this procedure comes from [3], Section 3.

The proofs of the following consequences of Theorem 5.3 are essestially reformulations of the proofs given in [3].

5.5 Corollary. *Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X. Then the following statements are equivalent*:

(i) *$\{U_t\}_{t\in\mathbb{R}}$ is uniformly continuous*;

(ii) $\|B\| < +\infty$;

(iii) *there exists $b \in X$, $0 < 1/\|B\| \leqslant b \leqslant 1$, such that*

$$U_t(x) = b^{it} x b^{-it}, \qquad x \in X, t \in \mathbb{R}.$$

Proof. The implication (iii) $\Rightarrow$ (i) is obvious and (i) $\Rightarrow$ (ii) follows from [15], Theorem 9.4.2.

Let us suppose now that (ii) is verified; then $X^B((0, \|B\|]) = X$, so, using Theorem 1.6(ii), $X^B([1/\|B\|, +\infty)) = X$. It follows that for $0 < \lambda < 1/\|B\|$ we have $X^B((0, \lambda]) = \{0\}$. Consequently H is simply invariant and

$$\begin{aligned} p_\lambda{}^H &\in X, && 0 \leqslant \lambda \leqslant \infty, \\ p_\lambda{}^H &= 0, && 0 \leqslant \lambda < 1/\|B\|, \\ p_\lambda{}^H &= 1, && 1 \leqslant \lambda \leqslant \infty. \end{aligned}$$

Using Theorem 5.3, (iii) follows with $b = \int_0^{+\infty} \lambda \, dp_\lambda{}^H$. Q.E.D.

Corollary 5.6 implies the well-known theorem of R.V. Kadison and S. Sakai asserting that each derivation of a von Neumann algebra is inner (see [3], Section 4), but also the converse implication is true (see [24], proof of Corollary 4.1.14). We still remark that Corollary 5.6 and the above statement about derivations remain valid also for AW^*-algebras (see [23]).

5.7. COROLLARY. *Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X.*
Then the following statements are equivalent:

(i) *there exists $d \in B(H)$, $d \geq 0$ and injective, such that*

$$U_t(x) = d^{it}xd^{-it}, \qquad x \in X,\ t \in \mathbb{R};$$

(ii) *H is simply invariant relative to $\{U_t\}_{t\in\mathbb{R}}$;*

(iii) *there exists $b \in X$, $0 \leq b \leq 1$, b injective, such that*

$$U_t(x) = b^{it}xb^{-it}, \qquad x \in X,\ t \in \mathbb{R}.$$

Proof. If (i) is verified, then by Theorem 5.3, $H = H^d((0, \|d\|))$ is simply invariant.

Suppose now that (ii) is verified. Since

$$p_\lambda^H \in X, \qquad 0 \leq \lambda \leq \infty,$$
$$p_\lambda^H = 1, \qquad 1 \leq \lambda \leq \infty,$$

and using again Theorem 5.3, (iii) results with $b = \int_0^{+\infty} \lambda \, dp_\lambda^H$.

Finally the implication (iii) $\Rightarrow$ (i) is trivial. Q.E.D.

The equivalence of (i) and (ii) in this corollary is a particular case of a theorem of Borchers [5].

It is easy to see that if $X \subset B(H)$ is a von Neumann algebra and $\{U_t\}$ is an X_*-continuous one-parameter group of $*$-automorphisms of X, then there exists a greatest projection p which belongs both to the center of X and to $X^B([1, 1])$ such that for some $b \in X$, $0 \leq b \leq p$, $s(b) = p$, we have

$$U_t(x) = b^{it}xb^{-it}, \qquad x \in X,\ t \in \mathbb{R}.$$

Using Corollary 5.7, it follows that $p = 1 - p_0^H$. In particular, the projection p_0^H does not depend on the spatial representation of X.

Now let X be a maximal commutative $*$-subalgebra of $B(H)$, that is $X = X'$, and suppose that $X \neq X^B([1, 1]) = \mathbb{C}$. Consider an invariant subspace K. By Corollary 5.4 $p_\infty^K - p_0^K$ is either 0 or 1, so K is either doubly invariant or simply invariant and with support 1.

In the same situation, p_0^H is either 0 or 1. If p_0^H would be 0, then by Corollary 5.7, we should have $X^B([1, 1]) = X$. Consequently $p_0^H = 1$.

Using Theorem 5.2, it results that the doubly invariant subspaces are exactly eH, $e \in X$ projection.

Consider now the cases:

$$X = \text{the von Neumann algebra of all multiplication operators with elements of } L^\infty(S^1) \text{ on } L^2(S^1),$$

$$(U_t f)(\zeta) = f(e^{-it}\zeta),$$

and

$$X = \text{the von Neumann algebra of all multiplication operators with elements of } L^\infty(\mathbb{R}) \text{ on } L^2(\mathbb{R}),$$

$$(U_t f)(s) = f(t - s).$$

Using the Stone–von Neumann–Mackey theorem (see [20, 21]) for the pairs of dual locally compact abelian groups $(S^1, \mathbb{Z})$, respectively, $(\mathbb{R}, \mathbb{R})$ as indicated in [14, Lecture V], it is easy to derive from Theorem 5.3 the Beurling–Helson–Lowdenslager theorem respectively the Lax–Helson–Lowdenslager theorem (see [14], Theorem 7).

We shall give now a maximality property of the spectral subspace $X^B((0, \lambda])$:

5.8. THEOREM. *Let $X \subset B(H)$ be a von Neumann algebra, $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, K a simply invariant subspace with support* 1 *relative to $\{U_t\}_{t\in\mathbb{R}}$ and $0 < \lambda < +\infty$.*

Then

$$X^B((0, \lambda]) = \{x \in X; xK_\mu \subset K_{\lambda\mu} \text{ for all } 0 < \mu < +\infty\}.$$

Proof. By Theorem 5.3 there exists a strongly continuous one-parameter group $\{u_t\}$ of unitaries on H such that, denoting its analytic generator by b, we have:

$$U_t(x) = u_t x u_{-t}, \qquad x \in X,\ t \in \mathbb{R},$$

$$H^b((0, \mu]) = K_\mu, \qquad 0 < \mu < +\infty.$$

Denote by

$$Y = \{x \in X; xK_\mu \subset K_{\lambda\mu} \text{ for all } 0 < \mu < +\infty\}.$$

Then Y is an X_*-closed linear subspace of X which contains $X^B((0, \lambda])$. If $x \in X$, then for any $t \in \mathbb{R}$

$$U_t(x) K_\mu = u_t x u_{-t} K_\mu = u_t x K_\mu \subset u_t K_{\lambda\mu} = K_{\lambda\mu}, \qquad 0 < \mu < +\infty,$$

so

$$U_t(x) \in Y.$$

Let $\lambda < \nu < +\infty$ and x be an element of the spectral subspace of $\{U_t \mid Y\}$ corresponding to $[\nu, +\infty)$. Then $x \in X^B([\nu, +\infty))$, so, by Theorem 1.6, $x^* \in X^B((0, 1/\nu])$. It follows that

$$x^*K_\mu \subset K_{(1/\nu)\mu}, \qquad 0 < \mu < +\infty,$$

consequently

$$x^*xK_\mu \subset K_{(\lambda/\nu)\mu}, \qquad 0 < \mu + \infty.$$

Let $n \geq 1$; for each polynomial P without free term, we have:

$$P((x^*x)^n)\, K_\mu \subset K_{(\lambda/\nu)^n\mu}, \qquad 0 < \mu < +\infty.$$

Since the support of $(x^*x)^n$ can be approximated in the strong operatorial topology by operators of the form $P((x^*x)^n)$, with P as above, it follows that

$$s(x^*x)\, K_\mu = s((x^*x)^n)\, K_\mu \subset K_{(\lambda/\nu)^n\mu}, \qquad 0 < \mu < +\infty.$$

Consequently,

$$s(x^*x)\, K_\mu \subset \bigcap_{n=1}^{\infty} K_{(\lambda/\nu)^n\mu} = K_0 = \{0\}, \qquad 0 < \mu < +\infty.$$

Since $\bigcup_{0<\mu<+\infty} K_\mu$ is dense in H, it follows that $s(x^*x) = 0$, that is $x = 0$.

Using Corollary 3.2, we deduce that Y coincides with the spectral subspace of $\{U_t \mid Y\}$ corresponding to $(0, \lambda]$. In particular, $Y \subset X^B((0, \lambda])$ and in conclusion, $Y = X^B((0, \lambda])$. Q.E.D.

It is clear that if X is commutative, then

$$X^B((0, \lambda]) = \{x \in X;\, xK \subset K_\lambda\}, \qquad 0 < \lambda < +\infty;$$

in particular,

$$X^B((0, 1]) = \{x \in X;\, xK \subset K\}.$$

Theorem 5.8 can be used to prove the following extension of some classical maximality results:

5.9. COROLLARY. *Let X be a commutative C^*-algebra, $\mathscr{F}$ a Banach space in duality with X, and $\{U_t\}_{t\in\mathbb{R}}$ an $\mathscr{F}$-continuous one-parameter group of $\mathscr{F}$-continuous $*$-automorphisms of X. Suppose that the spectral subspace of $\{(U_t^* \mid \mathscr{F}^{B^*})^*\}_{t\in\mathbb{R}}$ corresponding to $[1, 1]$ consists from the scalar multiples of the unity and is not equal to $(\mathscr{F}^{B^*})^*$. Then $X^B((0, 1])$ and $X^B([1, +\infty))$ are maximal elements of the family of all $\mathscr{F}$-closed subalgebras of X which are not equal to X.*

Proof. For convenience, we denote by $Y = (\mathscr{F}^{B*})^*$, $V_t = (U_t^* \mid \mathscr{F}^{B*})^*$, and $D =$ the analytic generator of $\{V_t\}$. Using Corollary 1.5, it is easy to see that the Y_*-closure of $\bigcup_{0<\lambda<1} \pi^B X^B((0, \lambda])$ contains $\bigcup_{0<\lambda<1} Y^D((0, \lambda])$.

By Theorem 1.6(jj), $X^B((0, 1]) = X \Leftrightarrow X^B([1, +\infty)) = X \Leftrightarrow X^B([1, 1]) = X \Leftrightarrow Y^D([1, 1]) = Y$. So $X^B((0, 1]) \neq X$.

Now let M be an $\mathscr{F}$-closed subalgebra of X, $M \supset X^B((0, 1])$, $M \neq X^B((0, 1])$, $M \neq X$, and let us find a contradiction.

Denote by $N = \mathbb{C} + Y_*$-closure of $\pi^B M$; then N is a Y_*-closed subalgebra of Y and by Corollary 3.2

$$N \supset \mathbb{C} + Y_*\text{-closure of} \bigcup_{0<\lambda<1} \pi^B X^B((0, \lambda])$$

$$\supset Y^D([1, 1]) + Y_*\text{-closure of} \bigcup_{0<\lambda<1} Y^D((0, \lambda]) = Y^D((0, 1]).$$

Suppose that $N = Y^D((0, 1])$. Using Corollary 1.5 and Theorem 4.1, it follows that $N = (\bigcup_{1<\lambda<+\infty} \mathscr{F}^{B*}([\lambda, +\infty)))_Y^{\perp}$, so $M \subset (\bigcup_{1<\lambda<+\infty} \mathscr{F}^{B*}([\lambda, +\infty)))_X^{\perp}$. Using again Corollary 1.5, we obtain $M \subset X^B((0, 1])$, in contradiction with the choice of M. Consequently, $N \neq Y^D((0, 1])$.

Suppose now that $N = Y$. Since M is $\mathscr{F}$-closed and $M \neq X$, there exists $\psi \in \mathscr{F}$, $\psi \neq 0$, which vanishes on M. Then ψ vanishes on $X^B((0, 1]) \subset M$ too, so, by Corollary 1.5 and Theorem 4.1, $\psi \in \mathscr{F}^{B*}([1, +\infty)) \subset \mathscr{F}^{B*} = Y_*$.

It follows that ψ vanishes on $\mathfrak{N} = Y_*$-closure on $\pi^B M$. It is clear that $\mathfrak{N}$ is a Y_*-closed subalgebra of Y and since $\mathbb{C} + \mathfrak{N} = N = Y$, it is also a two-sided ideal of Y. Following [24], Proposition 1.10.5, there exists a central projection p of Y such that $\mathfrak{N} = Yp$. Since $\psi \neq 0$, we have $\mathfrak{N} \neq Y$, so $p \neq 1$ and since $\mathbb{C} \neq Y$, we have also $p \neq 0$. On the other hand, as

$$\mathfrak{N} \supset Y_*\text{-closure of} \bigcup_{0<\lambda<1} \pi^B X^B((0, \lambda]) \supset \bigcup_{0<\lambda<1} Y^D((0, \lambda]),$$

by Theorem 1.6(ii), it results

$$\mathfrak{N} \supset \bigcup_{0<\lambda<1} Y^D((0, \lambda]) + \bigcup_{1<\lambda<+\infty} Y^D([\lambda, +\infty)).$$

Using Corollary 3.2, we obtain that

$$\mathfrak{N} = Y_*\text{-closure of} \Big(\bigcup_{0<\lambda<1} Y^D((0, \lambda]) + \bigcup_{1<\lambda<+\infty} Y^D([\lambda, +\infty)) \Big),$$

so $\mathfrak{N}$ is $\{V_t\}$-invariant. Hence $p \in Y^D([1, 1]) = \mathbb{C}$ and this means that either $p = 1$ or $p = 0$, both of which are impossible. Consequently, $N \neq Y$. Thus we have a Y_*-closed subalgebra N of Y, $N \supset Y^D((0, 1])$, $N \neq Y^D((0, 1])$, $N \neq Y$.

Let $\varphi \neq 0$ be a normal positive functional on Y. Then $e = \bigvee_{r \text{ rational}} U_r s(\varphi)$ is a nonzero projection in $Y^D([1, 1]) = \mathbb{C}$, so $e = 1$. It follows that there exists a faithful normal positive functional φ_0 on Y.

Since $N \neq Y$, by [24], Theorem 1.8.9, there exist $y \in Y$, a normal positive functional ρ on Y, and $\epsilon_0 > 0$ such that

$$\inf_{z \in N} \langle (y - z)^*(y - z), \rho \rangle \geq \epsilon_0 .$$

Then, obviously

$$\inf_{z \in N} \langle (y - z)^*(y - z), \varphi_0 + \rho \rangle \geq \epsilon_0 .$$

Denote by $\pi = \pi^{\varphi_0 + \rho}$, $H = H^{\varphi_0 + \rho}$, $\xi = \xi^{\varphi_0 + \rho}$. Since $\varphi_0 + \rho$ is faithful, π induces a $*$-isomorphism of Y onto the von Neumann algebra $\pi Y \subset B(H)$ and $(\pi Y)' = \pi Y$. The subspace

$$K = \bigcap_{1 < \lambda < +\infty} \overline{\pi(Y^D((0, \lambda])N)\xi}$$

is invariant relative to the one-parameter group of $*$-automorphisms of πY induced by $\{V_t\}$ via π, so it is either doubly invariant or simply invariant and with support 1.

Let us suppose first that K is doubly invariant. Then

$$\begin{aligned} K &= [\pi(Y^D((0, \tfrac{1}{2}]))K] \\ &\subset [\pi(Y^D((0, \tfrac{1}{2}])\, Y^D((0, 2])N)\xi] \\ &\subset [\pi(Y^D((0, 1])N)\xi] \\ &= \overline{\pi(N)\xi} \subset K, \\ K &= \overline{\pi(N)\xi}, \end{aligned}$$

and there exists a projection $f \in \pi Y$ with

$$K = fH.$$

Since $\xi \in K$, we have $f\xi = \xi$, so

$$\overline{\pi(N)\xi} = fH \supset \overline{f\pi(Y)\xi} = \overline{\pi(Y)f\xi} = \overline{\pi(Y)\xi} = H.$$

Consequently, there exists $z \in N$ such that

$$\langle (y - z)^* (y - z), \varphi_0 + \rho \rangle = \| \pi(y)\, \xi - \pi(z)\, \xi \|^2 < \epsilon_0 ,$$

which is impossible.

It follows that K is simply invariant and with support 1 and for all $z \in N$,

$$\pi(z) K \subset K.$$

Using the observation after Theorem 5.8, it follows that $N \subset Y^D((0, 1])$ which is also impossible.

The obtained contradiction shows that $X^B((0, 1])$ is maximal between all $\mathscr{F}$-closed subalgebras of X which are not equal to X.

Since $X^B([1, +\infty)) = X^B((0, 1])^*$, the same statement holds for $X^B([1, +\infty))$ too. Q.E.D.

It would be interesting to have noncommutative extensions of Corollary 5.9. We remark that the first part of the proof of Corollary 5.9, namely the reduction to the case when X is a W^*-algebra and $\{U_t\}$ is an X_*-continuous group of $*$-automorphisms, is true also when X is not commutative.

Corollary 5.9 implies immediately Wermer's Maximality Theorem (see [16, p. 93]) and the fact that $H^\infty(S^1)$ is a maximal $*$-closed subalgebra of $L^\infty(S^1)$ (see [16, p. 194, Corollary; 14, Lecture IV, Section 2]). Similar statements can be obtained in the cases $X = C_0(\mathbb{R})$, $\mathscr{F} = M(\mathbb{R})$, $(U_t f)(s) = f(s-t)$ and $X = L^\infty(\mathbb{R})$, $\mathscr{F} = L^1(\mathbb{R})$, $(U_t f)(s) = f(s-t)$.

Finally we shall give another application of the procedure to improve implementing groups sketched before Corollary 5.6.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and $\{U_t\}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. We say that $\lambda \in (0, +\infty)$ belongs to the *spectrum* $\sigma(U)$ *of* $\{U_t\}$ if for each $\delta > 1$, the spectral subspace $X^B([\lambda\delta^{-1}, \lambda\delta])$ contains nonzero elements. It is easy to see that $\sigma(U)$ is a closed subset of $(0, +\infty)$.

The above definition of the spectrum of $\{U_t\}$ is a reformulation of the definition given in [3] : $\lambda \in (0, +\infty)$ belongs to the above-defined $\sigma(U)$ if and only if $\ln \lambda$ belongs to the spectrum of $\{U_t\}$ as it is defined in [3].

There exists a connection between the periodicity properties of $\{U_t\}$ and its spectrum:

5.10. LEMMA. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\mathbb{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, $t_0 > 0$, and $\lambda_0 = e^{(2\pi/t_0)}$.*

Then the following statements are equivalent:

(i) $U_{t_0} = 1$;

(ii) $\sigma(U) \subset \{\lambda_0^n; n \in \mathbb{Z}\}$.

Proof. Suppose that (i) is verified.

Let $\lambda \in (0, +\infty)\setminus\{\lambda_0^n; n \in \mathbb{Z}\}$. Then there exists $n_\lambda \in \mathbb{Z}$ such that $\lambda_0^{n_\lambda+1} < \lambda < \lambda_0^{n_\lambda}$. Let $\epsilon > 0$ be such that $\lambda_0^{n_\lambda+1} < \lambda e^{-3\epsilon}$ and $\lambda e^{3\epsilon} < \lambda_0^{n_\lambda}$.

Consider an arbitrary element $x \in X^B([\lambda e^{-\epsilon}, \lambda e^{\epsilon}])$ and let $f \in L^1(\mathbb{R})$ be such that $\hat{f} \in C^2(\mathbb{R}), \hat{f} = 1$ on $[\ln \lambda - 2\epsilon, \ln \lambda + 2\epsilon]$, and

$$\operatorname{supp} \hat{f} \subset [\ln \lambda - 3\epsilon, \ln \lambda + 3\epsilon]$$
$$\subset ((n_\lambda - 1) \ln \lambda_0, n_\lambda \ln \lambda_0) = \left(\frac{2(n_\lambda - 1)\pi}{t_0}, \frac{2n_\lambda \pi}{t_0} \right).$$

By [8], Corollary 5.7,

$$x = \mathscr{F} - \int_{-\infty}^{+\infty} f(t)\, U_t x \, dt.$$

Since $s \to 1 - e^{it_0 s}$ does not vanishe on $\operatorname{supp} \hat{f}$, $s \to (1/(1 - e^{it_0 s}))\hat{f}(s)$ is a well-defined function with compact support belonging to $C^2(\mathbb{R})$.

Consequently, there exists $g \in L^1(\mathbb{R})$ such that $\hat{f}(s) = (1 - e^{it_0 s})\, \hat{g}(s)$, that is $f(t) = g(t) - g(t - t_0)$. Using (i), it follows that

$$x = \mathscr{F} - \int_{-\infty}^{+\infty} (g(t) - g(t - t_0))\, U_t x \, dt$$
$$= \mathscr{F} - \int_{-\infty}^{+\infty} g(t)(U_t x - U_{t+t_0} x)\, dt = 0.$$

We conclude that $X^B([\lambda e^{-\epsilon}, \lambda e^{\epsilon}]) = \{0\}$, so $\lambda \notin \sigma(U)$. Hence (ii) is verified.

Suppose now that (ii) is verified and let $n \in \mathbb{Z}$. For each $x \in X^B([\lambda_0{}^n, \lambda_0{}^n])$, by [8], Property (iv) of spectral subspaces and [8], Corollary 4.6, we have

$$U_{t_0} x = \lambda_0^{it_0 n} x = e^{2n\pi i} x = x.$$

Let further $\ell, m \in \mathbb{Z}, \ell \leq m$. Using (ii) we obtain:

$$X^B([\lambda_0{}^\ell, \lambda_0{}^m]) = \sum_{n=\ell}^{m} X^B([\lambda_0{}^n, \lambda_0{}^n]).$$

Consequently, for each $x \in X^B([\lambda_0{}^\ell, \lambda_0{}^m])$ we have

$$U_{t_0} x = x.$$

Since $\bigcup_{\ell \leq m} X^B([\lambda_0{}^\ell, \lambda_0{}^m])$ is $\mathscr{F}$-dense in X, it follows that (i) is satisfied.

Q.E.D.

We shall prove now a particular property of minimal implementing groups.

5.11. Lemma. *Let $X \subset B(H)$ be a von Neumann algebra and $\{U_t\}_{t \in \mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X.*

If $\{u_t\}_{t \in \mathbb{R}}$ is a minimal implementing group with support 1 for $\{U_t\}_{t \in \mathbb{R}}$ then

$$\sigma(u) \subset \sigma(U).$$

Proof. Let b be the analytic generator of $\{u_t\}$ and $K = H^b((0, 1])$. Then

$$H^b((0, \lambda]) = \bigcap_{\lambda < \mu < +\infty} [X^B((0, \mu])K], \qquad 0 < \lambda < +\infty.$$

Let $\lambda \in (0, +\infty)\backslash\sigma(U)$. Then there exists $\delta > 1$ such that for $\lambda\delta^{-1} < \mu_1 \leqslant \mu_2 < \lambda\delta$ we have $X^B((0, \mu_1]) = X^B((0, \mu_2])$. Then for $\lambda\delta^{-1} < \lambda_1 \leqslant \lambda_2 < \lambda\delta$ we have $H^b((0, \lambda_1]) = H^b((0, \lambda_2])$. Consequently $\lambda \notin \sigma(u)$. Q.E.D.

Using Lemmas 5.10 and 5.11, we can prove an implementation result analogous to Corollaries 5.6 and 5.7:

5.12. THEOREM. *Let $X \subset B(H)$ be a von Neumann algebra, $\{U_t\}_{t\in\mathbb{R}}$ an X_*-continuous one-parameter group of $*$-automorphisms of X, and $t_0 > 0$.*
Then the following statements are equivalent:

(i) $U_{t_0} = 1$ *and there exists a strongly continuous one-parameter group $\{v_t\}_{t\in\mathbb{R}}$ of unitaries on H such that*

$$U_t(x) = v_t x v_{-t}, \qquad x \in X, t \in \mathbb{R};$$

(ii) $U_{t_0} = 1$ *and there exists a simply invariant subspace with support 1 relative to $\{U_t\}_{t\in\mathbb{R}}$;*

(iii) *there exists a strongly continuous one-parameter group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries on H such that $u_{t_0} = 1$ and*

$$U_t(x) = u_t x u_{-t}, \qquad x \in X, t \in \mathbb{R}.$$

Proof. The implication (i) $\Rightarrow$ (ii) is a consequence of Corollary 5.5.

Suppose that (ii) is verified; by Theorem 5.3, there exists a minimal implementing group $\{u_t\}$ with support 1 for $\{U_t\}$. Let $\lambda_0 = e^{-(2\Pi/t_0)}$.

By Lemma 5.10, $\sigma(U) \subset \{\lambda_0^n; n \in \mathbb{Z}\}$ and by Lemma 5.11 $\sigma(u) \subset \sigma(U)$.

Consequently, $\sigma(u) \subset \{\lambda_0^n; n \in \mathbb{Z}\}$.

Using again Lemma 5.10, it results that $u_{t_0} = 1$. Thus (iii) holds. Finally, the implication (iii) $\Rightarrow$ (i) is trivial. Q.E.D.

It is possible that Theorem 5.12 could be used in a treatment of Connes invariant Γ (see [10], second paragraph).

We remark that in general $\sigma(U) \subset \sigma(B)$ and in many cases $\sigma(B) =$ the closure of $\sigma(U)$ in $[0, +\infty)$.

However, in [11] it is proved that if $X = L^\infty(\mathbb{R})$, $\mathscr{F} = L^1(\mathbb{R})$, and $(U_t f)(s) = f(s - t)$, then $\sigma(B) = \mathbb{C}$, so $\sigma(B) \neq$ the closure of $\sigma(U)$ in $[0, +\infty)$. The same statement holds for $X = L^\infty(S^1)$, $\mathscr{F} = L^1(S^1)$, and $(U_t f)(\zeta) = f(e^{-it}\zeta)$. These topics will be presented somewhere else.

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